RUNNING FOR THE EXIT: DISTRESSED SELLING AND ENDOGENOUS CORRELATION IN FINANCIAL MARKETS

Rama Cont; Lakshithe Wagalath

(Paris VI University)

Paper presented at the

9th International Paris Finance Meeting

December 20, 2011

www.eurofidai.org/december2011.html

Organization: Eurofidai & AFFI
Running for the Exit: 
distressed selling and endogenous correlation 
in financial markets

Rama Cont  
CNRS – Université Pierre & Marie Curie  
& Columbia University  
E-mail: rama.cont@upmc.fr

Lakshitha Wagalath  
Université Pierre & Marie Curie  
E-mail: lakshitha.wagalath@upmc.fr

January 2011

Key words  risk management, correlations, liquidity, short selling, predatory trading, feedback effects.

Abstract

We propose a simple multiperiod model of price impact from trading in a market with multiple assets, which illustrates how feedback effects due to distressed selling and short selling lead to endogenous correlations between asset classes. We show that distressed selling by investors exiting a fund and short selling of the fund’s positions by traders may have non-negligible impact on the realized correlations between returns of assets held by the fund. These feedback effects may lead to positive realized correlations between fundamentally uncorrelated assets, as well as an increase in correlations across all asset classes and in the fund’s volatility which is exacerbated in scenarios in which the fund undergoes large losses. By studying the diffusion limit of our discrete time model, we obtain analytical expressions for the realized covariance and show that the realized covariance may be decomposed as the sum of a fundamental covariance and a liquidity-dependent ‘excess’ covariance. Finally, we examine the impact of these feedback effects on the volatility of other funds. Our results provide insight into the nature of spikes in correlation associated with the failure or liquidation of large funds.
1 Introduction

Correlations in asset returns are a crucial ingredient for quantifying the risk of financial portfolios and a key input for asset allocation and trading. Correlations and covariances between returns of assets, indices and funds are routinely estimated from historical data and used by market participants as inputs for trading, portfolio optimization and risk management. Whereas sophisticated models—featuring stochastic volatility, conditional heteroskedasticity and jumps—have been proposed for univariate price dynamics, the dependence structure of returns is typically assumed to be stationary, either through a time-invariant correlation matrix or a copula, and estimated from historical time series of returns. For example, a popular method is to use (exponentially-weighted) moving average (EWMA) estimators of realized correlation.

On the other hand, empirical evidence points to high variability in realized correlations and model-based estimators of correlation [15]: these estimators exhibit large spikes or dips associated to market events. Figure 1 show examples of variability in time of realized correlations in equity indices; we observe a sharp increase in realized correlations associated with the collapse of Lehman Brothers on September 15th, 2008. More generally, unexpected correlation spikes are often associated with the liquidation of large positions by market participants. For instance, in 1998, due to heavy losses in its investments in Russian bonds, LTCM was forced to liquidate its positions after a sudden increase in the correlations across its—previously uncorrelated—positions which led to a sharp increase in its volatility [23]. Unexpected spikes of correlation arose between asset classes that used to be uncorrelated (Russian bonds and US equity for instance), leading to its collapse. A more complex phenomenon occurred in August 2007: between August 7 and August 9 2007, all long-short equity market neutral hedge funds lost around 20% per day whereas major equity indices hardly moved. Khandani and Lo [20] suggest that this ‘quant event’ of August 2007 was due to the unwinding of a large long-short market neutral hedge fund’s positions, that created extreme volatility on other funds with similar portfolios, while leaving index funds unaffected. These examples illustrate that "asset correlations can be different during a liquidity crisis because price movements are caused by distressed selling and predatory trading rather than fundamental news" [7].

The evidence for time-variation in the dependence structure of asset returns has motivated the development of new classes of stochastic models with time-dependent correlation structures [14, 13, 17, 25] in which the conditional distribution of asset returns is given by a multivariate distribution with a randomly evolving covariance structure whose evolution is specified exogenously. However, such models where correlation is represented as an exogenous risk factor fail to explain the presence of spikes in correlations associated with market events such as the liquidation of large funds. The examples cited above suggest the existence of an endogenous component in asset correlations, which should be modeled by taking into account the impact of supply and demand generated by investors, in particular in situations of market distress. Such endogenous variations in volatility and correlations, generated by systematic patterns in supply and demand linked to rule-based trading strategies, short selling or fire sales, have played an important role
in past financial crises and have been the focus of several studies [1, 2, 6, 8, 9, 22, 24] which underline the link between liquidity and volatility in financial markets.

Our contribution is to show that the intuitive link between distressed selling and endogenous changes can be modeled quantitatively in a rather simple, analytically tractable framework which allows to quantify the endogenous risk generated by fire sales, when investors facing losses simultaneously try to exit a fund.

1.1 Summary

We consider a fund investing in various asset classes/strategies whose returns are decomposed into random components that represent exogenous economic factors (fundamentals) and a term representing the price impact of sellers, which is a function of aggregate excess demand for each asset generated either by investors liquidating their positions or by speculators shorting the fund’s position once the fund value drops below a threshold. Simulations of this discrete-time model reveal that, even in the case of assets with zero fundamental correlation, one observes a significant positive level of realized correlation resulting in higher than expected fund volatility. Furthermore, this realized correlation is observed to be path-dependent.

We confirm the generic nature of these simulation results by studying the continuous-time limit of our model. We exhibit conditions under which the discrete time model exhibits a diffusion limit and provide explicit expressions for the instantaneous covariance, correlation across assets and fund volatility for the limiting diffusion process. Our analytical results show that realized covariance is the sum of a fundamental covariance and a excess covariance term which is path-dependent and varies inversely with market liquidity. Furthermore, this excess covariance is computable in our model setting. Even in absence of correlations between fundamentals, asset returns may exhibit significant positive correlation, resulting in higher fund volatility. We show that, even when market depth is constant, the liquidation of large fund positions can generate significant positive correlation between the fund’s assets, and may also generate spillover effects, affecting
the volatility of other funds holding similar assets. All these effects are shown to be analytical computable and expressions are given for their magnitude.

Our results point to the limits of diversification, previously discussed by many authors, but also allow one to quantify these limits. We show that a fund manager investing in apparently uncorrelated strategies may experience significant realized correlation across his/her strategies in the case of distressed selling by investors facing losses, thus losing the benefit of diversification exactly when it is needed. These results provide simple explanations for the sudden rise in correlations associated with the failure of LTCM in 1998 and the hedge fund losses of August 2007. Our study provides insight into the nature of spikes in correlation and fund volatility associated with the failure or liquidation of large funds and gives a quantitative framework to evaluate strategy crowding as a risk factor. In particular, the model explains how, in August 2007, the liquidation of a large long-short equity market neutral fund generated high volatility for funds with similar allocations while leaving index funds unaffected.

1.2 Related Literature

Empirical evidence of distressed selling and its impact on market dynamics has been documented by several previous studies. Funds experiencing large outflows sell their holdings, as documented by Coval and Stafford [12]. For regulatory reasons, after large losses, banks must sell risky assets, as discussed by Berndt et al. [5] for the corporate debt market. Khandani and Lo [20] describe how the need to reduce risk exposure compelled market-neutral long-short equity hedge funds to liquidate large position in equity markets in the second week of August 2007, generating a series of huge losses which are explained quantitatively by our model. Empirical studies by Jones et al. [10] show the importance of short selling in financial markets (40.2% and 39.2% of total dollar volume on the NYSE and Nasdaq, respectively). Haruvy and Noussair [18] examine empirically the effects of short selling restrictions finding that relaxing short selling constraints does not induce prices to track fundamentals. Our study provides a quantitative framework for analyzing these empirical observations.

Various theoretical models have been proposed for analyzing feedback effects resulting from fire sales in financial markets, mostly in a single-asset framework. Avellaneda and Lipkin [4] show how short selling in a single asset generates price anomalies and higher price volatility and violation of Call-Put parity. Market losses in subprime mortgage-backed securities, largely seen as being uncorrelated with equity markets, led to huge falls in equity markets as explained by Brunnermeier [6]. Shin [24] describes the mechanisms which amplified the recent financial crisis and the systemic risk they generate. Investors 'running for the exit' can generate spirals in prices and spillovers to other asset classes as well as a crowding effect, as discussed by Pedersen [22]. Andrade et al. [3] show how trading imbalances in one asset class can lead to deviation of prices from fundamental value in other asset classes. Short selling by predators is described in [7], where Brunnermeier shows how shorting the portfolio of a fund approaching its liquidation value can lead to the collapse of the fund. Our detailed quantitative analysis confirms these predictions. Whereas these studies mainly focus on asset prices and
fund value, our multi-asset framework allows for a computation of the impact of fund liquidation or short selling on realized correlation between assets and fund volatility.

1.3 Outline

The paper is organized as follows. Section 2 presents a multiperiod, multi-asset model of trading with price impact and introduces a simple model for distressed selling. Section 3 displays the results of the simulations of this model. In Section 4, we find the continuous-time limit of our discrete-time dynamics. Section 5 gives analytical expressions for the realized variance and covariance of asset returns in the continuous-time limit and uses these expressions to study the path-dependence of realized correlations and role of market depth. Using these analytical results, we show in Section 6 how feedback effects lead to endogenous volatility in a distressed fund and spillover effects across funds. Section 7 concludes.

2 A multi-asset model of price impact from distressed selling

Consider a market with \( n \) financial assets in which trading takes place at dates \( t_k = k\tau \) \((\tau = \frac{T}{M} \text{ and } 0 \leq k \leq M)\). The price of asset \( i \) at date \( t_k \) is denoted \( S^i_k \) and we denote \( S_k = (S^1_k, ..., S^n_k) \). It is useful, in the examples, to think of \( S^i \) as the value of an index or ETF representing a sector, asset class or geographic zone. At each period, the value of the assets is affected by exogenous economic factors, represented by an IID sequence \( \xi_k = (\xi^1_k, ..., \xi^n_k)_{1 \leq k \leq M} \) of centered random variables with covariance matrix \( \Sigma \).

In absence of other effects, the return of asset \( i \) at period \( k \) would be \( \tau m^i + \sqrt{\tau} \xi^i_{k+1} \) where \( m = (m_1, ..., m_n) \) are the expected returns in absence of price impact. We denote

\[
(S^i_{k+1})^* = S^i_k (1 + \tau m^i + \sqrt{\tau} \xi^i_{k+1}).
\]

We consider a large leveraged fund holding \( \alpha_i \geq 0 \) units of asset \( i \) with \( 1 \leq i \leq n \) between dates \( t = 0 \) and \( T \). Thus, between \( t_k \) and \( t_{k+1} \), exogenous economic factors move the value of the fund from \( V_k = \sum_{i=1}^{n} \alpha_i S^i_k \) to

\[
V^*_k = \sum_{i=1}^{n} \alpha_i (S^i_{k+1})^* = V_k + \sum_{i=1}^{n} \alpha_i S^i_k (\tau m^i + \sqrt{\tau} \xi^i_{k+1}).
\]

Investors enter the fund at \( t = 0 \) when the fund is valued at \( V_0 \). Like most investors in mutual funds, investors in the fund adopt a passive, buy and hold behavior as long as the fund is performing well. If the fund value drops below a threshold \( \beta_0 V_0 < V_0 \), investors progressively may exit their positions, generating a negative demand across all assets held by the fund, proportionally to the positions held by the fund. Our purpose is to model the price impact of this *distressed selling* and investigate its effect on realized volatility and correlations of the assets held by the fund.

We model the supply/demand pattern generated by distressed selling by introducing a function \( f : \mathbb{R} \to \mathbb{R} \) which measures the rate at which investors in the fund exit
their positions: when fund value drops from $V_k$ to $V^*_k$, investors redeem a fraction $f\left(\frac{V_k}{V_0}\right) - f\left(\frac{V^*_k}{V_0}\right)$ of their position in the fund. Thus, the net supply in asset $i$ due to distressed selling (or short selling) is equal to

$$-\alpha_i(f\left(\frac{V^*_k}{V_0}\right) - f\left(\frac{V_k}{V_0}\right))$$

The above assumptions on investor behavior imply that $f : \mathbb{R} \to \mathbb{R}$ is increasing, constant on $[\beta_0, +\infty]$. We furthermore assume that the fund is liquidated when the value reaches $\beta_{\text{liq}}V_0$ where $\beta_{\text{liq}} < \beta_0$. In practice, as the fund loses value and approaches liquidation, distressed selling becomes more intense: this feature is captured by choosing $f$ to be concave. Figure 2 gives an example of such a function $f$.

![Figure 2: Net supply due to distressed selling and short selling is equal to $-\alpha_i(f(\frac{V^*_k}{V_0}) - f(\frac{V_k}{V_0}))$](image)

It is well documented that sale of large quantities of assets impacts prices. Empirical studies [21, 11] provide evidence for approximate linearity of this price impact at daily and intraday frequencies; see also [19]. Between $t_k$ and $t_{k+1}$, given the net supply generated by short sellers and distressed sellers, market impact on asset $i$’s return is equal to

$$\frac{\alpha_i}{\lambda_i}(f\left(\frac{V^*_k}{V_0}\right) - f\left(\frac{V_k}{V_0}\right))$$

where $\lambda_i$ represents the depth of the market in asset $i$: a net demand of $\frac{\lambda_i}{100}$ shares for security $i$ moves the price of $i$ by one percent. Obizhaeva [21] studies empirically the link
between market depth and average daily volume (ADV) on NYSE and NASDAQ stocks, finding that \( \frac{\text{ADV} \sqrt{250}}{\lambda_i \sigma_i} \) is close to 1. We will use this relation to pick realistic values for the size of a large fund’s positions \( \alpha_i \) in terms of the market depth \( \lambda_i \), in the examples of Section 3. The supply/demand pattern generated by these distressed sellers exiting the fund may be amplified by short sellers or predatory traders: the presence of short sellers may result in scenarios where a fraction \( > 1 \) of the fund is exited/liquidated. From our perspective, their effect on price dynamics is similar and we will not distinguish between distressed (e.g. long) sellers and short sellers.

\[
\text{Equations 1 and 2 show that } S_{k+1} \text{ depends only on its value at } t_k \text{ and on } \xi_{k+1}, \text{ which is independent events previous to } t_k. \text{ } S = (S^1, ..., S^n) \text{ is thus a Markov Chain.}
\]

Using the dimensionless variables \( \tilde{S}_k = (\tilde{S}^1_k, ..., \tilde{S}^n_k) \) and \( \tilde{V}_k = \frac{V_k}{V_0} \), we can rewrite 1–2 as

\[
\tilde{S}^i_{k+1} = \tilde{S}^i_k \left( 1 + \tau m_i + \sqrt{\tau} \xi^i_{k+1} + \frac{\alpha_i}{\lambda_i} \left( f(\tilde{V}_k) + \sum_{j=1}^{n} \frac{\alpha_j S^j_k}{V_0} (\tau m_j + \sqrt{\tau} \xi^j_{k+1}) \right) - f(\tilde{V}_k) \right)
\]

where

\[
\tilde{V}_k = \sum_{j=1}^{n} \frac{\alpha_j S^j_k}{V_0} \tilde{S}^j_k
\]

Hence the dynamics of \( \tilde{S}_k \) is entirely determined by

- the drift \( m \)
- the sequence \( (\xi_k) \) and its fundamental covariance matrix \( \Sigma \)
- the vector \( (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n}) \) which expresses the sizes of the fund’s positions in each asset relative to the asset’s market depth. This is a dimensionless measure of the size of positions, which is relevant for measuring market impact in case of liquidation.
- the dollar proportions \( (\frac{\alpha_1 S^1_0}{V_0}, ..., \frac{\alpha_n S^n_0}{V_0}) \) initially invested in each asset by the fund
- the function \( f \) which describes the supply generated by distressed/short selling
3 Numerical experiments

3.1 Simulation procedure

We perform a Monte Carlo simulation (10^6 independent scenarios) of the multiperiod model described above for a fund investing in two strategies/asset classes with zero fundamental correlation and volatilities respectively given by 30% and 20%. We assume that the volume held by the fund on each asset is of the order of 20 times average daily volume for the asset. In comparison, LTCM’s on–balance sheet assets totalled around $125 billion, which represented 250 times average daily volume on the S&P 500 in 1998. We assume that the fund initially invests the same amount in both assets and that distressed sellers can trade once a day. We use the following parameters in our simulations:

- $m=0$
- $\xi$ is normal and $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$ with $\sigma_1 = 30\% \text{ year}^{-\frac{1}{2}}$ and $\sigma_2 = 20\% \text{ year}^{-\frac{1}{2}}$
- $\frac{\alpha_1}{V_0} = \frac{\alpha_2}{V_0} = \frac{1}{10}$: the fund’s position on asset 1 (resp. on asset 2) is equal to 10% of asset 1’s (resp. asset 2’s) market depth, or, using [21], around 15 times average daily volume for asset 1 (resp. around 20 times average daily volume for asset 2)
- $\frac{\alpha_1 S_0}{V_0} = \frac{\alpha_2 S_0}{V_0} = \frac{1}{2}$: the fund initially invests the same amount in 1 and 2
- We use the following choice for $f$: $f(x) = \left(\beta_{\text{liq}} - \beta_0\right)x^{\frac{4}{3}}$ which satisfies the conditions described in Section 2, with $\beta_0 = 0.95$ and $\beta_{\text{liq}} = 0.55$.

3.2 Realized variance and realized correlations

In each simulated path, we compute the log-returns of asset $r_i^k = \log(\frac{S_{i,k+1}}{S_{i,k}})$ for $i = 1, 2$.

Let $\overline{r_i}$ be the sample average of those returns: $\overline{r_i} = \frac{1}{M} \sum_{k=0}^{M-1} r_i^k$. For each sample path, we compute the realized covariance between assets $i$ and $j$:

$$\hat{C}_{i,j} = \frac{1}{T} \sum_{k=0}^{M-1} (r_i^k - \overline{r_i})(r_j^k - \overline{r_j})$$

and the realized correlation between $i$ and $j$: $\frac{\hat{C}_{i,j}}{\sqrt{\hat{C}_{i,i}\hat{C}_{i,j}}}$. The realized volatility for $i$ is given by $(\hat{C}_{i,i})^{\frac{1}{2}}$.

Figure 3 shows the distribution of the one-year realized correlation for the two strategies. In each scenario, we also computed realized correlation without feedback effects. Figure 4 is a scatter plot of the one-year realized correlation with and without feedback effects from distressed selling/short selling. Each point of the graph corresponds to one
trajectory (for clarity, we choose to display only 1000 trajectories on scatter plots). For each point of the graph and hence each trajectory, realized correlation in the presence of feedback effects (resp. without feedback effects) can be read on the vertical axis (resp. the horizontal axis).

![Figure 3: Distribution of realized correlation between the two securities (with $\rho = 0$) with and without feedback effects due to distressed selling](image1)

![Figure 4: Scatter plot of realized correlation with and without feedback effects due to distressed selling (each data point represents one simulated scenario)](image2)

In the presence of distressed selling, the distribution of realized correlation is significantly modified. Our simulations show that distressed selling by investors exiting funds with similar portfolios and short selling can generate significant realized correlation, even between assets with zero fundamental correlation. In Figure 3, the distribution of realized correlation without feedback effects reflects the statistical error in the estimation of correlation. Hence, the aspect of the distribution of realized correlation with feedback effects due to distressed selling or short selling reflects the effects of such trading on correlation between assets: average correlation in the presence of feedback effects is higher than its fundamental value $\rho = 0$ and the profile of its distribution presents a thick upper tail. In Figure 4, all points are above the Y=X axis, confirming the fact that distressed selling increases correlation between assets. In the presence of feedback effects, correlation becomes path dependant. It is interesting to examine the distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$, triggering distressed selling/short selling.

**Conditional correlation:** In Figure 5, we divide trajectories into two categories, whether fund value reaches $\beta_0 V_0$ between 0 and T or not and we display the distribution of realized correlation for those two categories: in plain line, the distribution of realized correlation
in scenarios where fund value reaches $\beta_0 V_0$, triggering distressed selling; in dotted line, the distribution of realized correlation in scenarios where fund value remains above $\beta_0 V_0$ and there is no distressed selling or short selling. Realized correlation conditional on the fact that distressed selling took place is significantly higher than realized correlation in scenarios where there was no distressed selling. In scenarios where distressed selling took place, price impact affects all assets of the fund in the same direction during the time the fund’s market value is below the threshold $\beta_0 V_0$. This results in higher realized correlation in those scenarios.

![Figure 5: Distribution of realized correlation in scenarios where fund value reaches $\beta_0 V_0$ between 0 and $T$ (plain line) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted line)](image)

Asset volatility: In the presence of feedback effects from distressed selling/short selling, asset volatility increases. Figure 6 shows that the distribution of realized volatility of each asset, in scenarios where there was distressed selling, is centered around a higher value than the asset’s fundamental volatility and presents a thick upper tail. In such scenarios, assets are more volatile than in scenarios without distressed selling. The action of distressed sellers (and short sellers) increases the amplitude of price moves and generates higher asset volatility. This should result in higher fund volatility.

### 3.3 Fund volatility

Figure 7 is a scatter plot of fund volatility, with and without feedback effects from distressed selling/short selling. We also compare the distribution of fund volatility in
Figure 6: Distribution of realized volatilities for each security in scenarios where fund value reaches $\beta_0 V_0$ between 0 and $T$ (plain lines) and in scenarios where fund value remains above $\beta_0 V_0$ (dotted lines) (with $\sigma_1 = 30\%$ and $\sigma_2 = 20\%$)

scenarios where the fund reaches $\beta_0 V_0$ or not. Figure 8 displays the distributions of fund volatility in those two scenarios.

Distressed selling increases the fund’s volatility: the distribution of realized fund volatility presents a thick upper tail when there are feedback effects from short sellers or distressed sellers. Figure 7 underlines the fact that feedback effects increase the fund’s volatility. Figure 8 shows that when there is distressed selling, the fund is more volatile than when fund value remains above $\beta_0 V_0$ and there is no distressed selling.

Our simulations show that, even in the case of assets with zero fundamental correlation, one observes a significant positive level of realized correlation resulting in higher than expected fund volatility.

4 Diffusion limit

To confirm that the phenomena observed in the numerical experiments are not restricted to particular parameter choices or a particular choice of the function $f$, we will now analyze the continuous-time limit of our discrete-time model: the study of this limit allows one to obtain analytical formulas for realized correlation which confirm quantitatively the effects observed in the numerical experiments.

Our main theoretical result is the following theorem which describes the diffusion limit of the price process.
Theorem 4.1 Under the assumption that $E(|\xi|^4) < \infty$ and that $f \in C^3_0$ such that $\sup |xf'(x)| < \min \frac{\alpha_i}{\lambda_i}$, $S_{[\tau]}$ converges weakly to a diffusion $P_t = (P^1_t, \ldots, P^n_t)$ when $\tau$ goes to 0 where

$$\frac{dP^i_t}{P^i_t} = \mu_i(P_t)dt + (\sigma(P_t)dW_t)_i \quad 1 \leq i \leq n$$

where $\mu$ (resp., $\sigma$) is a $\mathbb{R}^n$-valued (resp. matrix-valued) adapted process defined by

$$\mu_i(P_t) = m_i + \frac{\alpha_i}{2\lambda_i} f'' \left( \frac{V_t}{V_0} \right) \frac{\pi_t \Sigma \pi_t}{V_0^2}$$

$$\sigma_{i,j}(P_t) = A_{i,j} + \frac{\alpha_i}{\lambda_i} f' \left( \frac{V_t}{V_0} \right) \left( A \pi_t \right)_j$$

Here $W_t$ is an n-dimensional Brownian motion, $\pi_t = (\alpha_1 P^1_t, \ldots, \alpha_n P^n_t)$ is the (dollar) allocation of the fund, $V_t = \sum_{1 \leq k \leq n} \alpha_k P^k_t$ the value of the fund and $A$ is a square-root of the fundamental covariance matrix: $AA^t = \Sigma$.

Proof In order to study the continuous-time limit of the Markov chain $S$, we first show (see Appendix) that, under the assumption that $E(|\xi|^4) < \infty$ and that $f \in C^3_0$ such that $\sup |xf'(x)| < \min \frac{\alpha_i}{\lambda_i}$, we have:
Lemma 4.2 For all \( \epsilon > 0, 1 \leq i, j \leq n \) and \( r > 0 \):

\[
\lim_{\tau \to 0} \sup_{\|S\| \leq r} \frac{1}{\tau} \mathbb{P}(|S_{k+1} - S_k| \geq \epsilon |S_k = S) = 0
\]

\[
\lim_{\tau \to 0} \sup_{\|S\| \leq r} \left( \frac{1}{\tau} \mathbb{E}(S_{k+1} - S_k | S = S) - b_i(S) \right) = 0
\]

\[
\lim_{\tau \to 0} \sup_{\|S\| \leq r} \left( \frac{1}{\tau} \mathbb{E}[(S_{k+1} - S_k^i)(S_{k+1} - S_k^j) | S = S] - a_{i,j}(S) \right) = 0
\]

where \( a : \mathbb{R}^n \to \text{Sym}_n(\mathbb{R}) \) and \( b : \mathbb{R}^n \to \mathbb{R}^n \) are given by

\[
a_{i,j}(S) = S^i S^j \left( \Sigma_{i,j} + \frac{\alpha_i}{\lambda_j} f' \left( \frac{<\alpha_i S >}{V_0} \right) \sum_{1 \leq l \leq n} \frac{\alpha_l}{V_0} \Sigma_{i,l} + \frac{\alpha_i}{\lambda_i} f' \left( \frac{<\alpha_i S >}{V_0} \right) \sum_{1 \leq l \leq n} \frac{\alpha_l}{V_0} \Sigma_{i,l} \right)
\]

\[
+ S^i S^j \left( \frac{\alpha_i \alpha_j}{\lambda_i \lambda_j} f'' \left( \frac{<\alpha_i S >}{V_0} \right) \sum_{1 \leq l, p \leq n} \frac{\alpha_l \alpha_p}{V_0^2} \Sigma_{i,l} \Sigma_{j,p} \right)
\]

\[
b_i(S) = S^i \left( m_i + \frac{\alpha_i}{2 \lambda_i} f''' \left( \frac{<\alpha_i S >}{V_0} \right) \sum_{1 \leq j, l \leq n} \frac{\alpha_j \alpha_l}{V_0^3} \Sigma_{j,l} \right)
\]

Define the differential operator \( G : C^\infty_0(\mathbb{R}^n) \to C^1_0(\mathbb{R}^n) \) by

\[
G h(x) = \frac{1}{2} \sum_{1 \leq i, j \leq n} a_{i,j}(x) \partial_i \partial_j h + \sum_{1 \leq i \leq n} b_i(x) \partial_i h
\]

Let \( \sigma_{i,j}(S) = A_{i,j} + \frac{\alpha_i}{\lambda_j} f' \left( \frac{<\alpha_i S >}{V_0} \right) \sum_{1 \leq k \leq n} \frac{\alpha_k}{V_0} S^k A_{k,j} \), where \( AA^t = \Sigma \). \( a_{i,j}(S) = S^i S^j (\sigma \sigma^t)_{i,j}(S) \), so \( a \) is a symmetric non-negative function. Under the assumption that \( \sup |xf'(x)| < \min \frac{1}{m} \) (see appendix), \( a \) is invertible. As \( f' \) and \( f'' \) are \( C^0_0 \), \( a \) and \( b \) are Lipschitz-continuous functions.

So, by [16, Theorem 4.2, Ch.7], when \( \tau \to 0 \), \( S(\tau) \) converges in distribution to \((\mathbb{P}, (P_t)_{t \in [0,T]})\) the unique solution of the stochastic differential equation

\[
dP_t = b(P_t)dt + a(P_t)dW_t
\]

* When market depth is infinite (i.e. price impact is negligible) the continuous-time limit is a multivariate geometric Brownian motion and the covariance of the log-returns is given by the ‘fundamental’ covariance: \( \text{cov}(\ln P^i_t, \ln P^j_t) = t \Sigma_{ij} \).

* As \( f \) is concave on \([\beta_{01}, \beta_0]\), \( f'' \) is negative and the action of distressed sellers and short sellers pushes down the price as investors exit the fund i.e. when \( \frac{V_0}{V_0} \in [\beta_{01}, \beta_0] \).

* The expression of \( \sigma \) shows that distressed selling modifies correlation between assets, asset volatility and fund volatility. We will focus on this phenomenon in the next sections.
5 Realized correlations

5.1 Realized covariance

The following result follows by direct computation from Theorem 4.1:

**Proposition 5.1** The realized covariance (quadratic covariation) of $\ln S^i, \ln S^j$ on $[0, t]$ is

$$\frac{1}{t} \int_0^t C_{i,j}^s ds$$

where the instantaneous covariance $C_{i,j}^s$ is given by

$$C_s = \Sigma + \frac{1}{V_0'} f'(\frac{V_s}{V_0}) [\Lambda_\pi_t \Sigma + \Sigma_\pi_s \Lambda_t^t] + \frac{1}{V_0'} (f')^2(\frac{V_s}{V_0}) < \pi_s, \Sigma_\pi_s > \Lambda \Lambda^t$$

with

- $\pi_t = (\alpha_1 P_1 t, ..., \alpha_n P_n t)^t$ denotes the (dollar) holdings of the reference fund
- $\Lambda = (\frac{\alpha_1}{\lambda_1}, ..., \frac{\alpha_n}{\lambda_n})^t$ represents the positions of the reference fund in each market as a fraction of the respective market depth.

The expression for the realized covariance of asset returns shows that realized covariance is the sum of the fundamental covariance and an excess covariance term which is path-dependent and varies inversely with market depth. Excess covariance depends on the ratio $\frac{\alpha}{\lambda}$, which compares the positions of the fund to the market depth in each asset. When market depth is infinite, instantaneous covariance reduces to fundamental covariance. Moreover, the expression of instantaneous covariance shows that $C_{i,j}^t$ is a deterministic and continuous function of vector $\pi_t$, hence the impact of distressed selling on realized correlation is computable in this setting. Realized covariance and correlation between assets depend on the derivative of $f$, which represents the rate at which investors exit their positions when the fund underperforms.

In scenarios where the fund value stays above $\beta_0 V_0$ realized covariances converge to their fundamental value. However, as soon as the fund value falls below the threshold $\beta_0 V_0$ which triggers distressed selling, excess covariance appears: in such distress scenarios, realized correlation and realized variance differ from the values implied by the ‘fundamental covariance’ $\Sigma$. In the case where fundamental correlation is positive between all pairs of assets, distressed selling increases realized covariance. As shown by Eq. (5.1), the magnitude of this effect is determined by the size $\alpha_i$ of the positions being liquidated relative to the depth $\lambda_i$ of the market in these assets: this is further discussed in Section 5.4. It is also interesting to notice that when the fund invests significantly in an asset $i$ compared to its market depth and when fund value drops below $\beta_0 V_0$, instantaneous covariance between $i$ and any other asset $j$ in the market is different from its fundamental value $\Sigma_{i,j}$ (as $\frac{\alpha_i}{\lambda_i} f'(\frac{V_s}{V_0}) (\Sigma_{i,j}) \neq 0$).

5.2 Case of zero fundamental correlations

We now focus on the case of a diagonal covariance matrix $\Sigma$ ($\Sigma_{i,j} = 0$ for $i \neq j$): the $n$ assets are uncorrelated. We denote $\Sigma_{i,i} = \sigma_i^2$ ($\sigma_i$ is asset $i$’s volatility).
Corollary 5.2 If the fundamental covariance matrix $\Sigma$ is diagonal, then, for all $1 \leq i, j \leq n$, the instantaneous covariances are given by

$$C_{t \tau}^{i,j} = \frac{\alpha_i}{\lambda_i} f'(V_t) \frac{\alpha_j}{V_0} P_t^i \sigma_t^2 + \frac{\alpha_i}{\lambda_i} f'(V_t) \frac{\alpha_i}{V_0} P_t^i \sigma_t^2 + \frac{\alpha_i \alpha_j}{\lambda_i \lambda_j} (f')^2(V_t) \sum_{1 \leq l \leq n} \left( \frac{\alpha_l P_l^i \sigma_l}{V_0} \right)^2 \geq 0$$

and

$$C_{t \tau}^{i,i} = \sigma_i^2 + 2 \frac{\alpha_i}{\lambda_i} f'(V_t) \frac{\alpha_i}{V_0} P_t^i \sigma_t^2 + \frac{\alpha_i}{\lambda_i} (f')^2(V_t) \sum_{1 \leq l \leq n} \left( \frac{\alpha_l P_l^i \sigma_l}{V_0} \right)^2 \geq \sigma_i^2$$

Realized correlation between $i$ and $j$ (resp. realized variance for asset $i$) between 0 and $T$ are equal to $\frac{\int_0^T C_{t \tau}^{i,j} \, dt}{(\int_0^T C_{t \tau}^{i,i} \, dt)^{1/2}}$ (resp. $(\int_0^T C_{t \tau}^{i,i} \, dt)^{1/2}$).

Since $f$ is increasing, realized correlations are positive and volatility of asset $i$ is greater than $\sigma_i$: in absence of fundamental correlation, distressed selling generates positive realized correlation across the fund’s strategies and increase the volatility of all assets detained by the fund. This is due to the fact that when $V_t < \beta_0 V_0$, all strategies owned by the fund face a net demand of the same sign. In particular a large fall in fund value generates a negative demand by investors across all positions held by the fund and increases the amplitude of price movements. These results confirm our simulations.

Even if the fund invests in 'fundamentally' uncorrelated strategies, in scenarios where the fund experiences losses e.g. $V_t < \beta_0 V_0$ and approaches liquidation, distressed selling by investors leads to a positive realized correlation between the fund’s strategies, reducing the benefit of diversification.

To check whether these asymptotic results are relevant in the case of daily rebalancing, we compare the theoretical formula for realized covariance in continuous time given in Corollary 5.2 and the realized covariance in a discrete-time market as calculated in 3.2. Figure 9 shows that the higher the trading frequency, the better the concordance between empirical realized covariance (calculated as in section 3.2) and the continuous-time result (given by Corollary 5.2). More precisely, a linear regression of the realized covariance with respect to the theoretical values computed using Corollary 5.2 shows good agreement between the empirical and theoretical values: the regression yields a slope of 0.95 ($R^2 = 0.63$) for $\frac{T}{M} = \frac{1}{250}$ and a slope of 0.99 ($R^2 = 0.96$) for $\frac{T}{M} = \frac{1}{2500}$.

5.3 The path-dependent nature of realized correlation

Proposition 5.1 shows that instantaneous covariance $C_{t \tau}^{i,j}$ is a deterministic function of $\pi_t = (\alpha_1 P_t^1, \ldots, \alpha_n P_t^n)^T$. Figure 10 shows an example of the evolution of correlation, given a trajectory $\pi_t$. We used the same parameters as in our simulations (section 3) and we display the evolution of $\frac{V_t}{V_0}$ on the graph at the left and the evolution of realized correlation $\frac{1}{T} \int_0^T \frac{C_{t \tau}^{1,2,ds}}{(\int_0^T C_{t \tau}^{1,1,ds}(\int_0^T C_{t \tau}^{2,2,ds})^2)$, in the figure on the right. We see that as long as fund value stays above $\beta_0 V_0$, realized correlation is equal to 0. Losses greater
than this threshold generate distressed selling and lead to a positive *endogenous* correlation between the asset returns. As shown in Figure 10, this excess correlation is path-dependent: it depends on the performance of the fund. Fund losses are amplified by investors exiting funds with similar allocations or by those trading against the fund.
This not only drives down the fund value but increases the correlation between its two strategies to unexpected levels. As a result, realized correlation among strategies can be much higher than the 'fundamental' correlation, exactly when the fund is in dire need of the relief promised by diversification. The spiral can be triggered by a large loss in one of its strategies. This leads to investors exiting similar funds, others shorting its positions and thus generates a high correlation among all its positions.

5.4 Liquidation impact

Theorem 4.1 and Proposition 5.1 show that the price dynamics and correlation between assets are functions of the positions of the fund relative to the market depth of each asset: \( \Lambda = (\frac{\alpha_1}{\lambda_1}, ... , \frac{\alpha_n}{\lambda_n})^t \). These results suggest that \( \frac{\alpha_i}{\lambda_i} \) may be used as an indicator of the impact on asset \( i \) of the liquidation of the fund’s position. When \( \frac{\alpha_i}{\lambda_i} \to 0 \), we find, as expected, a Black-Scholes model with constant correlation between assets. Proposition 5.1 shows that the excess covariance tends to 0 when market depth goes to infinity. Corollary 5.2 proves that in the case of assets with zero fundamental correlation, the bigger the fund’s positions compared to the market depth of each asset, the more correlated its strategies will be, as can be seen on Figure 11.

![Figure 11: Distribution of realized correlation for different values of \( \frac{\alpha}{\lambda} \)](image)

It is interesting to underline the fact that when all assets, except one, denoted \( i_0 \), have infinite market depths and when distressed selling/short selling occurred in the market (\( \exists t_0, \tilde{V}_{t_0} \in [\beta_{liq}, \beta_{pred}] \)), all strategies are positively correlated with strategy \( i_0 \) (as \( C_{t_0 i_0} > 0 \)). Figure 12 shows that when one asset (asset 2) has finite market depth, the underperformance of other assets (asset 1) leads to strictly positive realized correlation.
Endogenous risk and spillover effects

The computation of realized correlations is relevant for the assessment of the (realized) volatility of portfolios: the explicit formulas obtained in Section 5 allow one to quantify the impact of distressed selling on the volatility of the fund being exited/shorted, and see how distressed selling on one fund affects other funds’ volatility.

6.1 Realized variance of a fund subject to distressed selling

Proposition 6.1 The fund’s realized variance between 0 and \( t \) is equal to \( \frac{1}{t} \int_0^t \Gamma_s \, ds \) where \( \Gamma_s \), the instantaneous variance of the fund, is given by:

\[
\Gamma_s V_s^2 = \langle \pi_s, \Sigma \pi_s \rangle + \frac{2}{V_0} \int \left( \frac{V_s}{V_0} \right)^2 \langle \pi_s, \Sigma \pi_s \rangle \langle \Lambda, \pi_s \rangle + \frac{1}{V_0^2} \left( \int \left( \frac{V_s}{V_0} \right)^2 \langle \pi_s, \Sigma \pi_s \rangle \langle \Lambda, \pi_s \rangle \right)^2
\]

(5)

where \( \pi_s \) and \( \Lambda \) are defined in Proposition 5.1.

The proof is given in the appendix. We note that distressed selling increases the fund’s volatility:

\[
\Gamma_s V_s^2 \geq \langle \pi_s, \Sigma \pi_s \rangle
\]

The fund’s instantaneous variance is equal to its fundamental value \( \frac{1}{t} \langle \pi_s, \Sigma \pi_s \rangle \) plus a term of order one in \( \frac{\alpha}{\lambda} \langle \Lambda, \pi_s \rangle \) and a term of order two in \( \frac{\alpha^2}{\lambda^2} \langle \Lambda, \pi_s \rangle^2 \). In a market with infinite market depth, \( \Gamma_s \) is equal to its fundamental value. As in the case of
instantaneous covariance between assets, $\Gamma_s$ is a continuous and deterministic function of $\pi_s$ and is a superposition of two regimes: a fundamental regime and an excess volatility regime, that is exacerbated with illiquidity ($\lambda$ is small) or when the fund has big positions ($\alpha$ is high). Note that, even without liquidity drying up ($\lambda$ constant), feedback effects may significantly increase fund volatility when investors 'run for the exit', generating spikes in correlation, even in absence of predatory trading by short sellers.

6.2 Fund volatility in the case of zero fundamental correlations

**Corollary 6.2** If the fundamental covariance matrix $\Sigma$ is diagonal, the instantaneous variance of the fund value is given by

$$\Gamma_t = (1 + \sum_{1 \leq i \leq n} \frac{\alpha_i P_i}{V_0} \alpha_i f'(\frac{V_i}{V_0}))^2 \sum_{1 \leq j \leq n} (\frac{\alpha_j P_j}{V_j} \sigma_j)^2$$

**Proof** This is a consequence of Proposition 6.1 and the fact that $\Sigma$ is a diagonal matrix.

The fund’s realized variance between 0 and $T$ is equal to $\frac{1}{T} \int_0^T \Gamma_t \, dt$. Corollary 6.2 explains the observations of section 3 and confirms that, when the size of the fund’s positions are non-negligible with respect to market depth, distressed selling leads to an increase in fund volatility, even when the fund invests in assets with zero fundamental correlation. Similarly, fund volatility increases when similar funds liquidate part of their positions.

*These results point to the limits of diversification* when price impact is not negligible: even if the fund manager invests in uncorrelated strategies, short selling and liquidation by investors facing losses will correlate them positively, exactly in scenarios where the fund experiences difficulty, increasing the volatility of the portfolio and reducing the benefit of diversification. This may arise either because the fund has large positions or because other large funds are following similar strategies (strategy crowding). In the next section we explore this second situations, which may lead to contagion of losses across funds.

6.3 Spillover effects

We now examine the impact of distressed selling by investors in a large fund (called hereafter the reference fund) on the volatility of other funds.

Consider a (small) fund investing in the $n$ securities and following a self-financing strategy. We denote by $\mu_i^t$ the number of units of $i$ detained by the target fund at date $t$. Note that we allow for dynamic strategies. Its market value at $t$ is $M_t = \sum_{1 \leq i \leq n} \mu_i^t P_i^t$.

As the target fund’s strategy is self-financing, we have $dM_t = \sum_{1 \leq i \leq n} \mu_i^t dP_i^t$.

In our framework, the target fund’s strategy should impact prices and its action should modify the dynamics of $P$ given by Theorem 4.1. However, when the target
fund’s positions are very small compared to the size of the reference fund, the impact of its trading strategy is negligible compared to feedback effects due to distressed selling and short selling. As $M_t$ can take negative values, in order to quantify the risk of the portfolio $\mu$, we study $d[M]_t$ (and no longer $d[\log(M)]_t$).

Under the assumption that the size of the target fund is small, its strategy does not impact prices and $P$ still follows the dynamics given in Theorem 4.1 and we obtain

$$d[M]_t = <\pi^\mu_t, C_t \pi^\mu_t> dt,$$

where $\pi^\mu_t = (\mu^1_t P^1_t, ..., \mu^n_t P^n_t)^t$ and $C_t$ is the instantaneous covariance matrix of $P$ given in Proposition 5.1.

**Proposition 6.3** Assume that the target fund’s strategy does not impact prices. Then the quadratic variation of the fund value is given by

$$[M]_t = \int_0^t \gamma^M_s ds$$

where

$$\gamma^M_t = <\pi^\mu_t, \Sigma \pi^\mu_t> + \frac{2f'(\frac{V^t}{V_0})}{V_0} <\pi^\mu_t, \Pi \pi^\mu_t> + \frac{f'(\frac{V^t}{V_0})^2}{V_0^2} <\pi^\mu_t, \Pi \pi^\mu_t> (\langle \Lambda, \pi^\mu_t \rangle)^2$$

where

- $M_t = \sum_{1 \leq i \leq n} \mu^i_t P^i_t$ is the fund value
- $\pi^\alpha_t = (\alpha^1_t P^1_t, ..., \alpha^n_t P^n_t)$ denotes the (dollar) holdings of the reference fund,
- $\pi^\mu_t = (\mu^1_t P^1_t, ..., \mu^n_t P^n_t)$ denotes the (dollar) holdings of the target fund,
- $\Lambda = (\frac{\alpha^1_t}{\lambda^1}, ..., \frac{\alpha^n_t}{\lambda^n})$ represents the positions of the reference fund in each market as a fraction of the respective market depth.

This result shows how distressed selling in one fund affects the volatility of other funds. In the presence of feedback effects, the quadratic variation $[M]$ of the fund’s value is given by its fundamental value $\int_0^t <\pi^\alpha_t, \Sigma \pi^\alpha_t> ds$ plus excess volatility terms which correspond to the impact of distressed selling and depend on $\frac{\alpha}{\lambda}$.

It is interesting to note that this ‘contagion’ across portfolios depends on the similarity between the portfolio of the reference fund $\alpha$ and the target fund $\mu$. In particular, when the portfolios $\alpha$ and $\mu$ are orthogonal for $\Sigma (\langle \pi^\mu_t, \Sigma \pi^\alpha_t \rangle = 0)$, the term of order one in $\frac{\alpha}{\lambda}$ in Proposition 6.3 is zero and the target fund’s absolute variance is equal to its fundamental value plus a term of order two in $\frac{\alpha}{\lambda}$, whose magnitude is much smaller. More interestingly, if the allocations of the two funds verify the ‘orthogonality’ condition

$$\langle \Lambda, \pi^\mu_t \rangle = \sum_{1 \leq i \leq n} \frac{\alpha^i_t}{\lambda^i} \mu^i_t P^i_t = 0,$$  \hspace{1cm} (6)
distressed selling of investors in the reference fund does not affect the target fund’s variance:

\[ [M]_t = \int_0^t < \pi^\mu_s, \Sigma \pi^\mu_s > \, ds. \]

On the contrary, the excess volatility due to feedback is maximal when strategies \( \mu \) and \( \alpha \) are colinear (i.e. when the vectors \( \pi^\mu_t \) and \( \pi^\alpha_t \) are colinear).

These results shed some light on the ‘quant event’ of August 2007. In August 2007, long-short equity market-neutral funds experienced extreme volatility and large losses during three days, whereas there was no tangible effect on major equity indices in the same period. An explanation which has been advanced is that a large position in such a market-neutral long-short fund, was liquidated by an investor in this three day period. Our model suggests that this rapid liquidation would then exacerbate the volatility of other long-short market-neutral funds following similar strategies (i.e. whose allocation vector has a positive projection on the allocation vector of the fund being exited). Since, by construction of market-neutral funds, the holdings of index funds are orthogonal to market-neutral funds in the sense of the orthogonality condition (6), our model predicts that index funds would not be subject to these feedback effects: indeed, they were insensitive to this event.

Alternative explanations advanced for the August 2007 events are sometimes based on a supposed drying up of liquidity in equity markets during that period [20]. However, there is no evidence such a dry-up in liquidity occurred: in fact, trading in equity indices occurred seamlessly during this period. By contrast, the mechanism underlying our model does not require any time-varying liquidity for these effects to occur: indeed, all these effects are present even when the market depth \( \lambda_i \) is constant. Also, our explanation entails that the population of long-short market-neutral funds affected by this event had allocations with substantial ‘colinearity’ i.e. that “strategy crowding” was a major risk factor in this market. Our results show the relevance of strategy crowding as a risk factor and represent a first step in quantifying it. Our analysis points in particular to the necessity of using indicators based on the size of positions when quantifying crowding effects, via proxies such as the market capitalization of various strategies. Clearly, factors based on returns alone cannot capture such size effects.

7 Conclusion

We have presented a simple and analytically tractable model for investigating the impact of fire sales on volatility and correlation of assets held by a fund. Our model yields explicit results for the realized variance and realized correlations of assets held by the fund and shows that the realized covariance between returns of two assets may be decomposed into the sum of a ‘fundamental’ covariance and a liquidity-dependent ‘excess covariance’, which is found to be inversely proportional to the market depth of these assets.

We have shown that the presence of this excess covariance leads to endogenous risk for large portfolios – liquidating the positions of such a large portfolio entails a higher-than-expected volatility which may increase liquidation costs – as well as spillover effects:
distressed selling of investors in a large fund may also exacerbate the volatility of funds with similar allocations, while leaving funds verifying an ‘orthogonality’ condition unaffected. This underlines the necessity of considering ‘strategy crowding’ as a risk factor and gives a quantitative framework to evaluate such risk.

More generally, our study shows that “liquidity risk” and “correlation risk”, often treated as separate sources of risk, may be difficult to disentangle in practice: rather than being treated as an exogenous factor to be estimated using statistical methods, correlation risk needs to be modeled at its source, namely co-movements in supply and demand across asset classes. Each of these observations raises a point which merits an independent, in-depth study. We plan to pursue some of these research directions in a forthcoming work.

Appendices

A Proof of Theorem 4.1

Let us first show that $a$ is invertible. As $a_{i,j}(S) = S^i S^j (\sigma^i)_{i,j}(S)$, we show that $\sigma$ is invertible. Writing $\sigma$ as a matrix,

$$\sigma(S) = A + \frac{1}{V_0} f'(\frac{\alpha}{V_0}, S) \Lambda \pi(S) A = \left( I_d + \frac{1}{V_0} f'(\frac{\alpha}{V_0}, S) \Lambda \pi(S) \right) A$$

where $\pi(S) = (\alpha_1 S^1, ..., \alpha_n S^n)$, and using that $A$ is invertible, it suffices to show that the norm of the matrix valued process $S \mapsto \frac{1}{V_0} f'(\frac{\alpha}{V_0}, S) \Lambda \pi(S)$ is strictly below the norm of $S \mapsto S$. For an $n$ dimensional matrix $M$, we define the norm of $M$:

$$\|M\| = \max_i \sum_{j=1}^n |M_{i,j}|.$$ We then have $\|I_d\| = 1$. Furthermore, for all $S$, we have

$$\| f'(\frac{\alpha}{V_0}, S) \Lambda \pi(S) \| = \frac{1}{V_0} f'(\frac{\alpha}{V_0}, S) \max_i \frac{\alpha_i}{\lambda_i} \sum_{j=1}^n \alpha_j S^j$$

$$= \frac{\alpha}{V_0} S \max_i \frac{\alpha_i}{\lambda_i} \leq \sup_i \frac{\alpha_i}{\lambda_i} < 1$$

under our assumptions. We show the first part of Lemma 4.2 with less restrictive assumptions: $f \in C^1_b$ and there exists $\beta > 2$ such that $\mathbb{E}(\|\xi^\beta\|) < \infty$. We can write

$$|S_{k+1}^i - S_k^i| \leq |S_k^i| \left( \tau |m_i| + \sqrt{\tau} |\xi_{k+1}^i| + \frac{\alpha_i}{\lambda_i} \|f^\prime\|_\infty (\sum_{j=1}^n \frac{\alpha_j}{V_0} S_j^i (\tau |m_i| + \sqrt{\tau} |\xi_{k+1}^j|)) \right).$$

As a consequence, using $\mathbb{E}(\|\xi^\beta\|) < \infty$ and the convexity inequality yields

$$\left( \frac{1}{n} \sum_{1 \leq i \leq n} \gamma_i \right)^\beta \leq \frac{1}{n} \sum_{1 \leq i \leq n} \gamma_i^\beta$$
We focus on the second equation of Lemma 4.2. We have that

\[ f \]

We expand this product. Using the fact that \( f \)

Writing

Similarly,

\[ \sup_{\|S\| \leq r} P_k(|S^{q_i}_{k+1} - S^{q_i}_k| \geq \epsilon) \leq \sup_{\|S\| \leq r} \frac{E_k(|S^{q_i}_{k+1} - S^{q_i}_k|)}{\epsilon^\beta} \leq C\tau^\frac{4}{\epsilon^\beta} \]

for all \( \epsilon > 0 \). Finally, as \( \frac{\beta}{2} > 1 \), we find that

\[ \lim_{\tau \to 0} \sup_{\|S\| \leq r} \frac{1}{\tau} P_k(|S^{q_i}_{k+1} - S^{q_i}_k| \geq \epsilon) = 0. \]

We now focus on the second equation of Lemma 4.2. We have

\[ E(S^{q_i}_{k+1} - S^{q_i}_k | S_k = S) = E \left( S^i (\tau m_i + \sqrt{\tau} \xi^i_{k+1}) + \frac{\alpha_i}{\lambda_i} (f(\frac{V^{*}_{k+1}}{V_0}) - f(\frac{V_k}{V_0})) | S_k = S \right) \]

Writing \( f(\frac{V^{*}_{k+1}}{V_0}) - f(\frac{V_k}{V_0}) = \)

\[ f'(\frac{V_k}{V_0}) \left( \sum_{j=1}^{n} \frac{\alpha_j}{V_0} S^{i_j}_k (\tau m_j + \sqrt{\tau} \xi^j_{k+1}) \right) \]

and using the fact that \( f'' \) is bounded and \( E(|\xi|^3) < \infty \), we find that

\[ |E(S^{q_i}_{k+1} - S^{q_i}_k | S_k = S) - \tau b_i(S)| \leq C^2 \tau^\frac{4}{\epsilon^\beta}. \]

Similarly,

\[ E \left( (S^{q_i}_{k+1} - S^{q_i}_k)(S^{q_j}_{k+1} - S^{q_j}_k) | S_k = S \right). \]

We expand this product. Using the fact that \( f(\frac{V^{*}_{k+1}}{V_0}) - f(\frac{V_k}{V_0}) = \)

\[ f'(\frac{V_k}{V_0}) \left( \sum_{p=1}^{n} \frac{\alpha_p}{V_0} S^{p}_{k} (\tau m_p + \sqrt{\tau} \xi^p_{k+1}) \right) \]

that \( f' \) and \( f'' \) are bounded and that \( E(|\xi|^4) < \infty \), we obtain that:
\[ \mathbb{E} \left( \sqrt{\tau} e^{T_{k+1}} \sqrt{\tau} e^{T_{k+1}} | S_k = S \right) = \Sigma_{i,j} \tau \]

\[ \mathbb{E} \left( \sqrt{\tau} e^{T_{k+1}} \frac{\alpha_i}{\lambda_j} (f(V_{i+1}^t) - f(V_i^t)) | S_k = S \right) = \frac{\alpha_i}{\lambda_j} f' \left( \frac{\alpha_i S}{\tau} \right) \sum_{1 \leq i \leq n} \frac{\alpha_i}{\lambda_j} S^i \Sigma_{i,j} \tau + o(\tau) \]

\[ \mathbb{E} \left( \sqrt{\tau} e^{T_{k+1}} \frac{\alpha_i}{\lambda_j} (f(V_{i+1}^t) - f(V_i^t)) | S_k = S \right) = \frac{\alpha_i}{\lambda_j} f' \left( \frac{\alpha_i S}{\tau} \right) \sum_{1 \leq i \leq n} \frac{\alpha_i}{\lambda_j} S^i \Sigma_{i,j} \tau + o(\tau) \]

\[ \mathbb{E} \left( \sqrt{\tau} e^{T_{k+1}} \frac{\alpha_i}{\lambda_j} (f(V_{i+1}^t) - f(V_i^t)) | S_k = S \right) \]

where \( o(\tau) \) designates a term \( \tau \varepsilon(\tau) \) where \( \varepsilon \to 0 \) uniformly on every compact neighborhood of \( S \). All the other terms of the product involve the drift \( m \) and are also of order \( o(\tau) \). As a consequence, we obtain the last equation of Lemma 4.2.

### B Proof of Proposition 6.1

Theorem 4.1 gives us the dynamics of asset prices in the continuous-time limit:

\[ \frac{dP_i^t}{P_i^t} = \mu_i(P_t) dt + (\sigma(P_t) dW_i)_i \]

where \( W \) is an \( n \) dimensional Brownian motion, \( (\alpha_{i,j}(P_t) = A_{i,j} + \frac{\alpha_i}{\lambda_j} f' \left( \frac{V_i^t}{\tau} \right) \sum_{1 \leq k \leq n} \frac{\alpha_k}{\lambda_j} P_k^t A_{k,j} \) and \( \mu_i(P_t) = \frac{\alpha_i}{\lambda_j} f' \left( \frac{V_i^t}{\tau} \right) \sum_{1 \leq j \leq n} \frac{\alpha_j \alpha_i}{\lambda_j} P_i^t P_j^t \Sigma_{j,i} \). As a consequence, for all \( 1 \leq i \leq n \), \( P_i^t \) is strictly positive. We also have for all \( 1 \leq i \leq n \), \( \alpha_i > 0 \) and so: \( V_t = \sum_{1 \leq k \leq n} \alpha_k P_k^t \) is strictly positive. Let’s focus on the dynamics of the fund’s position.

\[ dV_t = \sum_{1 \leq i \leq n} \alpha_i dP_i^t = \sum_{1 \leq i \leq n} \alpha_i P_i^t (\mu_i(P_t) dt + (\sigma(P_t) dW_i)_i) \]

Dividing by \( V_t \) and denoting \( x_i^t = \frac{\alpha_i P_i^t}{V_t} \), we obtain

\[ \frac{dV_t}{V_t} = \sum_{1 \leq i \leq n} x_i^t (\mu_i(P_t) dt + (\sigma(P_t) dW_i)_i) \]

\[ = \sum_{1 \leq i \leq n} x_i^t \mu_i(P_t) dt + \sum_{1 \leq i \leq n} x_i^t (\sigma(P_t) dW_i)_i \]

\[ = \sum_{1 \leq i \leq n} x_i^t \mu_i(P_t) dt + \sum_{1 \leq i \leq n} x_i^t (\sum_{1 \leq j \leq n} \sigma_{i,j}(P_t) dW_j^i) \]
\[ = \sum_{1 \leq i \leq n} x_i^t \mu_i(P_t) dt + \sum_{1 \leq j \leq n} \sum_{1 \leq i \leq n} x_i^t \sigma_{i,j}(P_t) dW^j_t. \]

As a consequence, the instantaneous variance of the fund is \[ \sum_{1 \leq j \leq n} \left( \sum_{1 \leq i \leq n} x_i^t \sigma_{i,j}(P_t) \right)^2. \] The statement follows.

References


