# Capital Requirements and Asset Prices*

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## Abstract

We consider a pure-exchange general equilibrium economy populated by investors with heterogeneous preferences and beliefs. The investors receive labor incomes, which are not fully pledgeable, and can potentially default on their risky positions unless their asset holdings are collateralized. We study the equilibrium implications of a constraint that requires investors to keep their financial capital above a certain minimum level to provide sufficient collateral. We characterize periods in the economy in which mere possibility of a crisis makes constraints binding and significantly depresses interest rates and increases Sharpe ratios. We find that stock price-dividend ratios are higher in the constrained economy and the tightening of constraints emerges as a viable instrument for curbing asset volatilities in bad times. Our equilibrium is stationary, and both investors survive despite differences in beliefs. The equilibrium processes are derived in closed form.

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# 1. Introduction

Financial markets play a key role in facilitating risk sharing and efficient allocation of assets among investors. However, trading in financial assets often entails moral hazard due to investors' incentives to default on their risky positions. The moral hazard can be alleviated by collateralized trades whereby risky positions are guaranteed by financial capital that can be seized in the event of default. The latter arrangement restores the functionality of the financial system at a cost of reducing risk sharing among investors. Furthermore, requiring risky positions to be backed by financial capital affects asset prices by restricting the demand for risky positions. In this paper, we develop a parsimonious model which sheds light on the economic effects of such restrictions on asset prices, their moments, and the distribution of consumption and wealth in the economy. Our analysis is facilitated by closed-form solutions of the model and the stationarity of equilibrium.

We consider a Lucas (1978) economy with one consumption good populated by two representative investors with heterogeneous constant relative risk aversion (CRRA) preferences over consumption and heterogeneous beliefs about the growth rate of the aggregate consumption. The investors receive non-tradable labor incomes each period and invest their wealth in financial assets such as bonds and shares in the Lucas tree. Investor heterogeneity generates bilateral trades in assets and introduces time-variation in asset prices, their moments, and the amount of leverage. The bilateral trades are curbed by the fact that the investors can potentially default following financial losses, in which case, their financial assets can be seized by counterparties but their labor incomes cannot be fully expropriated. The default does not preclude the investors from re-entering the financial markets in the future. The arising moral hazard problem in the economy is resolved by requiring asset trades to be backed by collateral in such a way that the next-period's value of financial capital stays above a certain threshold at all times. We label the latter constraint as capital requirement. A special case of zero threshold means that the value of financial capital has to be positive at all times, so that all asset trades are cross-collateralized in such a way that losses on one asset are offset by the gains on the other assets. The latter requirement arises when the labor income is non-pledgeable, and hence, the investors cannot take risky positions backed by future labor income. A positive minimum capital requirement is analogous to constraints imposed on banks by regulators, whereas negative minimum capital requirement arises when part of the labor income can be pledged.

The aggregate consumption growth rates are independent and identically distributed (i.i.d.) but may occasionally experience large negative transitory shocks during low-



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probability production crises in the economy. These shocks allow us to explore how the mere anxiety about the possibility of a production crisis affects the economy by making capital requirement constraints binding. We solve the model in closed form for general risk aversions and beliefs and explore the effects of capital requirements on interest rates, Sharpe ratios, price-dividend ratios, stock return volatilities, and distributions of investors' consumption shares in the aggregate output. We summarize our main results below.

First, we study how the capital requirements affect interest rates and Sharpe ratios in the economy. Our analysis reveals that rare production crises and capital requirements amplify the effects of each other so that the interest rate is lower and the Sharpe ratio is higher than in an economy where either crises or capital requirements are absent. In particular, we show that during periods characterized by low wealth of the irrationally optimistic investor, mere possibility of a large drop in the aggregate consumption next period decreases the interest rates and increases Sharpe ratios relative to the unconstrained benchmark. These effects occur because investors "fly to quality" by buying riskless bonds when they expect constraints to bind next period. As a result, the interest rates drop and Sharpe ratios increase to compensate investors for holding stocks in bad times.

Next, we turn to asset prices and show that the capital requirement constraint increases stock price-dividend ratio relative to the unconstrained benchmark. The effects of constraints are stronger when investors are close to their default boundaries, which makes price-dividend ratios U-shaped functions of one of the investor's share of the aggregate consumption. Moreover, the price-dividend ratios become very sensitive and spike upwards in response to small economic shocks. In our economy, the consumption share of the pessimist is high in bad times, when aggregate consumption is low due to a series of bad shocks, and high in good times, when aggregate consumption is relatively high. Therefore, the U-shaped pattern implies that the price-dividend ratio is procyclical in good times, as in the data (e.g., Campbell and Shiller, 1988), but countercyclical in bad times.

Our explanation for the effect of capital requirements on asset prices is as follows. Absent any frictions, the investors' consumption shares, in general, vary from zero to one, so that the economic impact of an investor either vanishes completely or dominates the whole economy in the long-run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015, among others). The capital requirements impose limits on investors' financial losses. Therefore, their consumption shares stay within certain bounds which lie strictly between zero and one and are reached when investors hit their constraints. When an investor's constraint binds the other investor faces a loose constraint and has a consumption share that reaches its maximum and cannot increase any further, in contrast to the unconstrained economy. Therefore, the marginal utility of the latter investor is higher in the constrained than in the unconstrained economy. Consequently, the stock price is higher under capital requirements because the marginal utility of the investor with non-binding constraint is proportional to the state price density and can be used for pricing assets. The effect is stronger around times when investors' constraints bind. There are also two additional factors contributing to higher stock prices. First, capital requirements restrict the short-selling by pessimists, which inflates stock prices (e.g., Harrison and Kreps, 1978). Second, stock prices reflect the additional value due to the fact that stocks can be used as collateral in lieu of non-pledgeable labor income.

The dynamics of price-dividend ratio determines the effect of constraints on volatilities. We show that capital requirements dampen volatilities in bad times, when aggregate consumption is low, and amplify them in good times, when aggregate consumption is high. Therefore, capital requirements emerge as a useful tool for curbing excessive volatility in bad times. Intuitively, because the price-dividend ratio is procyclical in good and countercyclical in bad times, the price-dividend ratio and the dividend move in the same direction in good times and in opposite directions in bad times. Because the stock price is the product of the price-dividend ratio and the dividend, stock return volatility increases in good times and decreases in bad times. We find that the volatility exhibits clustering and is very sensitive to economic shocks when investors are close to hitting their constraints. In particular, the economy abruptly switches into periods of low, medium of high volatility.

Finally, we show that the investor heterogeneity leads to dynamic reallocations of wealth and consumption across the investors, which makes all equilibrium processes time-varying. We find that the distributions of investors' consumption shares are stationary because the capital requirements ensure that consumption shares never drop below certain levels, which serve as reflecting boundaries. We show that these distributions are well-defined and non-trivial, and characterize them in analytic form in terms of elementary functions. The stationarity of consumption shares provides an answer to a long-standing question posed by Friedman (1953) and further explored by Blume and Easly (2006) and others on whether irrational investors in financial markets can survive in the long-run. We show that in the presence of capital requirements all investors survive irrespective of their beliefs. Moreover, the distributions of investors' consumption shares can be bimodal, so that irrational investors can occasionally attain large consumption shares.

To our best knowledge, this paper is the fist to provide a tractable closed-form equilibrium processes and distributions of consumption shares in an economy with imperfect risk sharing among heterogeneous investors. The paper makes several methodological contributions. First, it develops a tractable way of inducing the stationary of equilibrium, and derives the stationary distributions in closed form for general risk aversions and beliefs. Second, it shows that in the economies with reflecting boundaries the equilibrium is characterized in terms of linear differential-difference equations of a peculiar type, where one of the terms has a delayed argument, which is additionally restricted to exceed the lower reflecting boundary. Third, the paper develops an efficient method for solving the latter differential-difference equations in closed form. Finally, the paper introduces a tractable discrete-time framework that facilitates the exposition in non-technical terms and permits passing to continuous-time limits. Due to its tractability and stationarity our model is a convenient benchmark for asset pricing research that can be extended in various directions.

Our paper contributes to a large literature in macroeconomics and finance that studies economies with constraints designed to avoid defaults. Closest to us are papers that study economies where investors have limited liability and face solvency constraints that require their trades to be collateralized. Deaton (1990) considers a partial equilibrium model in which investors trade in a riskless asset with an exogenous interest rate and solve their consumption choice problem subject to a non-negativity constraint on their financial wealth. Chien and Lustig (2010) study a similar constraint in an economy with a continuum of investors that receive non-pledgeable labor incomes affected by idiosyncratic shocks. Detemple and Serrat (2003) also study non-negative wealth constraint in a model where investors have heterogeneous beliefs but identical risk aversions. They mainly focus on interest rates and Sharpe ratios. The constraint studied in these works is a special case of ours. Moreover, we focus on heterogeneity in preferences and beliefs, derive new economic implications for asset prices and their moments, and find the equilibrium in closed form. Geanakoplos (2003, 2009) and Fostel and Geanakoplos (2014) develop a theory of collateral constraints with endogenous margins. Fostel and Geanakoplos (2008) apply this theory to study how leverage cycles cause contagions and flight to collateral.

Kehoe and Levine (1993) solve investors' optimization subject to a participation constraint under which investors are weakly better off not defaulting, and investors are permanently excluded form securities markets is they default. Kocherlakota (1996) derives a sub-game perfect equilibrium in the economy without commitment. Alvarez and Jermann (2000) demonstrate that participation constraints can be implemented by imposing certain "not too tight" solvency portfolio constraints. Alvarez and Jermann (2001) study quantitative implications of these constraints and demonstrate that they help explain equity premia and Sharpe ratios in the U.S. economy. In contrast to this literature, our investors have limited liability and are allowed to re-enter the market after default.

The paper is also related to the literature that studies economic effects of borrowing, margin, short-sale and position limit constraints (e.g., Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Gromb and Vayanos, 2002, 2010; Pavlova and Rigobon, 2008; Brunnermeier and Pedersen, 2009; Gârleanu and Pedersen, 2011; Buss et al, 2013; Chabakauri, 2013, 2015; Rytchkov, 2014, among others), portfolio insurance (e.g., Basak, 2005) and VaR constraints (e.g., Basak and Shapiro, 2001). The paper also relates to macro-finance and financial intermediation literatures that study economies with frictions (Kiyotaki and Moore, 1997; Brunnermeier and Sannikov, 2014; He and Krishnamurthy, 2012, 2013; Kondor and Vayanos, 2015). In particular, our constraint on financial wealth is similar to non-negative consumption constraints imposed on risk-neutral investors in Brunnermeier and Sannikov (2014) and Kondor and Vayanos (2015).

The unconstrained benchmark economy with heterogeneous preferences and beliefs is a special case of our economy. Therefore, the paper also contributes to large literature that studies frictionless economies with heterogeneous investors (e.g., Basak, 2005; Chan and Kogan, 2002; Yan, 2008; Longstaff and Wang, 2012; Bhamra and Uppal, 2014; Gârleanu and Panageas, 2014; Borovička, 2015, among others).

The paper is organized as follows. Section 2 discusses the economic setup and defines the equilibrium. Section 3 provides a characterization of equilibrium both in discrete-time and continuous-time economies. Section 4 provides the analysis of equilibrium and Section 5 concludes. Appendix provides the proofs.

## 2. Economic setup

We consider a pure-exchange infinite-horizon economy with one consumption good produced by an exogenous Lucas (1978) tree, and two representative heterogeneous investors $A$ and $B$ that hold shares in the tree and receive labor income each period. To facilitate the exposition, we start our analysis with a discrete-time economy with dates $t = 0, \Delta t, 2\Delta t, \ldots$, and later take a continuous-time limit.

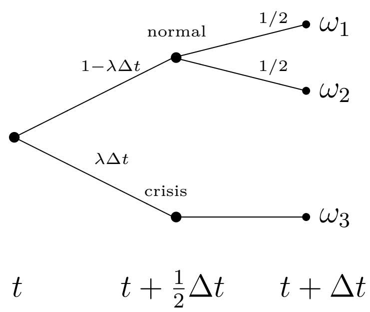


**Figure 1**
**States of the Economy**
After time $t$ the economy moves to a normal state with probability $1-\lambda\Delta t$ and to a crisis state with probability $\lambda\Delta t$. Conditional on being in a normal state the economy moves to either $\omega_1$ or $\omega_2$ with equal probabilities.

## 2.1. Aggregate output and securities markets

At each point of time $t = 0, \Delta t, 2\Delta t, \ldots$ the economy is in one of the three states: $\omega_1$, $\omega_2$, and $\omega_3$. With probability $1-\lambda\Delta t$ the economy is either in state $\omega_1$ or state $\omega_2$, which we call *normal states*, and with probability $\lambda\Delta t$ it is in state $\omega_3$, which we call the *crisis state.* Parameter $\lambda$ is the *crisis intensity.* States $\omega_1$ and $\omega_2$ have probabilities 1/2 conditional on the economy being in a normal state. Figure 1 depicts the structure of uncertainty.

At date $t$ the tree produces $D_t\Delta t$ units of aggregate output, where $D_t$ follows a process

$$\Delta D_t = D_t[\mu_D \Delta t + \sigma_D \Delta w_t + J_D \Delta j_t], \tag{1}$$

where $\mu_D$, $\sigma_D$, and $J_D$ are constant expected output growth, output growth volatility, and a percentage drop in the aggregate output in the crisis state, respectively, and $\Delta D_t = D_{t+\Delta t} - D_t$ is the change in the aggregate output. Furthermore, $w_t$ and $j_t$ are discrete-time analogues of Brownian motion and Poisson processes, respectively.[1] Processes $w_t$ and $j_t$ follow dynamics $w_{t+\Delta t} = w_t + \Delta w_t$ and $j_{t+\Delta t} = j_t + \Delta j_t$, where increments $\Delta w_t$ and $\Delta j_t$ are independent and identically distributed random variables given by:

$$\Delta w_t = \begin{cases} +\sqrt{\Delta t}, & \text{in state } \omega_1, \\ -\sqrt{\Delta t}, & \text{in state } \omega_2, \\ 0, & \text{in state } \omega_3, \end{cases} \qquad \Delta j_t = \begin{cases} 0, & \text{in state } \omega_1, \\ 0, & \text{in state } \omega_2, \\ 1, & \text{in state } \omega_3. \end{cases} \tag{2}$$

It can be easily verified that $\mathbb{E}_t[\Delta w_t|\text{normal}] = 0$ and $\text{var}_t[\Delta w_t|\text{normal}] = \Delta t$, similar to a Brownian motion, where $\mathbb{E}_t[\cdot]$ and $\text{var}_t[\cdot]$ are expectation and variance conditional on

[1] Chabakauri (2014) considers a similar process for the aggregate consumption $D_t$ and demonstrates its convergence to a continuous-time Lévy process as $\Delta t \to 0$.

time-$t$ information, respectively. We assume that parameters $\mu_D$, $\sigma_D$, and $J_D$ are such that $D_t > 0$ at all times.

Fractions $l_A$ and $l_B$ of the aggregate output $D_t\Delta t$ are paid to investors $A$ and $B$ as their labor incomes, respectively, and fraction $1 - l_A - l_B$ is paid as a dividend to the shareholders. Labor incomes are non-tradable. The investors can trade three securities at date $t$: 1) a riskless bond in zero net supply which pays one unit of consumption at date $t + \Delta t$; 2) one stock in net supply of one unit, which is a claim to the stream of dividends $(1 - l_A - l_B)D_t\Delta t$ paid by the Lucas tree; 3) a zero net supply one-period insurance contract that pays one unit of consumption in the crisis state $\omega_3$ and zero otherwise.[2] Time-$t$ bond, stock, and insurance prices $B_t$, $S_t$, and $P_t$, respectively, are determined endogenously in equilibrium. Absent any frictions the market is complete.

## 2.2. Investor heterogeneity and optimization problems

The investors have heterogeneous CRRA preferences over consumption, given by

$$u_i(c) = \begin{cases} \dfrac{c^{1-\gamma_i}}{1-\gamma_i}, & \text{if } \ \gamma_i \neq 1, \\ \ln(c), & \text{if } \ \gamma_i = 1, \end{cases} \tag{3}$$

where $i = A, B$. Moreover, the investors agree on observed time-$t$ asset prices and the aggregate output but disagree on the probabilities of states. Investor $A$ is rational and has correct probabilities

$$\pi_A(\omega_1) = \frac{1 - \lambda\Delta t}{2}, \quad \pi_A(\omega_2) = \frac{1 - \lambda\Delta t}{2}, \quad \pi_A(\omega_3) = \lambda\Delta t, \tag{4}$$

whereas investor $B$ has biased probabilities

$$\pi_B(\omega_1) = \frac{1 - \lambda_B\Delta t}{2}(1 + \delta\sqrt{\Delta t}), \quad \pi_B(\omega_2) = \frac{1 - \lambda_B\Delta t}{2}(1 - \delta\sqrt{\Delta t}), \quad \pi_B(\omega_3) = \lambda_B\Delta t, \tag{5}$$

where crisis intensity $\lambda_B$ and disagreement parameter $\delta$ are such that probabilities $\pi_B(\omega)$ are positive. It is immediate to verify that $\pi_B(\omega_1) + \pi_B(\omega_2) + \pi_B(\omega_3) = 1$, and hence, $\pi_B(\omega)$ defines a valid probability measure. Throughout the paper by $\mathbb{E}_t^i[\cdot]$ and $\mathrm{var}_t^i[\cdot]$ we denote conditional expectations and variances under the probability measure of investor $i$.

[2]In reality, the role of our insurance contract is performed by CDS contracts. The contract in our setting is just an Arrow-Debreu security which is required to complete the underlying market (absent portfolio constraints), and can be replaced by any other non-redundant security such as an option.

The time-$t$ expected output growth rate conditional on being in a normal state under the beliefs of investor $B$ is given by:

$$\mathbb{E}_t^B\left[\frac{\Delta D_t}{D_t}\Big|\text{normal}\right] = (\mu_D + \delta\sigma_D)\Delta t, \tag{6}$$

Therefore, parameter $\delta$ measures the extent of the investor disagreement about the expected output growth during normal times. For tractability, we assume that investor $B$ does not update probabilities over time. We also assume that investor $B$ is less risk averse and more optimistic than investor $A$, so that $\gamma_A \geq \gamma_B$, $\lambda \geq \lambda_B$ and $\delta \geq 0$.

At date 0 the investors have certain endowments of financial assets. The total time-$t$ wealth of investor $i$ is given by $W_{it} + l_i D_t \Delta t$, where $W_{it}$ is the financial wealth, defined as the time-$t$ value of assets acquired at the previous date, and $l_i D_t \Delta t$ is labor income. At each point of time $t$, investor $i$ allocates total wealth to $c_{it}\Delta t$ units of consumption, $b_{it}$ units of bond, and a portfolio of risky assets $n_{it} = (n_{i,St}, n_{i,Pt})$, which consists of $n_{i,St}$ units of stock and $n_{i,Pt}$ units of insurance.

In a frictionless economy the financial wealth can become negative when investors purchase risky assets against their future labor income. However, we assume that labor incomes are not fully pledgeable and the investors can potentially default when their financial wealth becomes negative. The investors also have limited liability and can re-enter the market after default, which gives rise to a moral hazard problem, similar to the related literature (e.g., Chien and Lustig, 2010; Geanakoplos, 2009). As noted in Chien and Lustig (2010), the limited liability is equivalent to holding a call option written on financial wealth. The moral hazard problem is addressed here by requiring the investors to hold collateralized positions in riskless bonds and risky assets so that their financial capital stays above a certain minimum level at all times, as elaborated below.

The investors maximize their expected discounted utility with time-discount $\rho$

$$\max_{c_{it}, b_{it}, n_{it}} \mathbb{E}_t^i\left[\sum_{\tau=t}^{\infty} e^{-\rho\tau} u_i(c_{i\tau})\Delta t\right], \tag{7}$$

subject to the self-financing budget constraints, given by

$$W_{it} + l_i D_t \Delta t = c_{it}\Delta t + b_{it}B_t + n_{it}(S_t, P_t)^\top, \tag{8}$$

$$W_{i,t+\Delta t} = b_{it} + n_{it}\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t,\ \mathbf{1}_{\{\omega_{t+\Delta t}=\omega_3\}}\Big)^\top, \tag{9}$$

and the capital requirement constraint

$$W_{i,t+\Delta t} \geq k_i\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big), \tag{10}$$

where $i = A, B$, $\mathbf{1}_{\{\omega_{t+\Delta t}=\omega_3\}}$ is the one-period insurance payment in state $\omega_3$, and $k_i$ is the *tightness of the constraint*. By $c^*_{it}$, $b^*_{it}$, and $n^*_{it}$ we denote investors' optimal consumptions and asset holdings, and by $W^*_{it}$ their financial wealths under the optimal strategies.

The capital requirement constraint (10) has the following interpretations. When $k_i = 0$, the investors are required to fully collateralize their asset holdings in such a way that their next-period financial wealth is positive, and hence, losses on one asset are offset by gains on other assets. This constraint naturally arises when labor incomes are non-pledgeable. For $k_i = 0$ the constraint is the same as in Chien and Lustig (2010). However, while Chien and Lustig (2010) study the economies where investors differ only with respect to their idiosyncratic labor shocks, the investors in our economy differ in risk aversions and beliefs. Therefore, our results are new even for the case $k_i = 0$. In the economy with a single risky asset the constraint is also the same as in Geanakoplos (2009), and requires the investors to fully collateralize debt and short positions. The constraint is also similar to non-negative consumption requirements imposed on risk-neutral investors in Brunnermeier and Sannikov (2014) and Kondor and Vayanos (2015).

When $k_i < 0$, the financial wealth can be negative to a certain extent. Such a constraint arises when part of the labor income can be pledged. The unconstrained benchmark economy is a special case of ours when $k_i = -\infty$. Finally, when $k_i > 0$, the constraint can be interpreted as a minimum capital requirement constraint frequently imposed on banks by financial regulators. In the latter case, we require that $k_A + k_B < 1$ because otherwise the aggregate wealth of investors would exceed the value of the whole market in violation of the market clearing conditions.

We observe that capital requirement (10) is not a limit to arbitrage and allows investors to eliminate arbitrage opportunities because adding to the investor's portfolio an arbitrage strategy that requires zero initial wealth and delivers nonnegative payoffs does not violate the constraint. Furthermore, we note that minimum capital threshold $k_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t)$ on the right-hand side of constraint (10) can be interpreted as a fraction of the value of the future labor income if it were tradable. Intuitively, because labor incomes and the dividend are both proportional to $D_{t+\Delta t}$, and $S_{t+\Delta t}$ is the price of the clam to future dividends $(1 - l_A - l_B)D_\tau\Delta t$, we observe that $l_i/(1 - l_A - l_A)(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t)$ can be interpreted as the value of future labor incomes $l_i D_\tau \Delta t$. The structure of the minimum capital is imposed for tractability.

## 2.3. Equilibrium

**Definition.** *An equilibrium is a set of asset prices* $\{B_t, S_t, P_t\}$ *and of consumption and portfolio policies* $\{c^*_{it}, b^*_{it}, n^*_{it}\}_{i\in\{A,\,B\}}$ *that solve optimization problem* (7) *for each investor, given processes* $\{B_t, S_t, P_t\}$*, and consumption and securities markets clear, that is,*

$$c^*_{At} + c^*_{Bt} = D_t, \quad b^*_{At} + b^*_{Bt} = 0, \quad n^*_{A,St} + n^*_{B,St} = 1, \quad n^*_{A,Pt} + n^*_{B,Pt} = 0. \tag{11}$$

In addition to asset prices, we derive price-dividend and wealth-aggregate consumption ratios $\Psi = S/\big((1 - l_A - l_B)D\big)$ and $\Phi_i = W^*_i/D$, respectively. We also derive annualized $\Delta t$-period riskless interest rates $r_t$, stock mean-returns $\mu_t$ and volatilities $\sigma_t$ in normal times, and the percentage change of the stock price in the crisis state, denoted by $J_t$.

Throughout the paper, we assume that the dividend processes are such that in the homogeneous-agent economies populated by only investors $A$ or investors $B$, respectively, the investors' value functions are finite, so that:

$$\left[\sum_{\tau=0}^{\infty} e^{-\rho\tau} u_i(D_\tau)\Delta t\right] < +\infty. \tag{12}$$

In Section 3 below, we provide easily verifiable conditions under which (12) is satisfied.

# 3. Characterization of equilibrium

In this section, we provide the characterization of equilibrium. We assume and later verify that the matrix of asset payoffs in all states is invertible, and hence, a hypothetical small unconstrained investor would face a complete-market economy. Moreover, the market is arbitrage-free, as discussed above. Consequently, for each investor's probability measure $\pi_i(\omega)$ there exists unique state price density (SPD) $\xi_{it}$ in the economy (e.g., Duffie, 2001) such that the asset prices satisfy the following equations:

$$B_t = \mathbb{E}^i_t\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\Big], \tag{13}$$

$$S_t = \mathbb{E}^i_t\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big)\Big], \tag{14}$$

$$P_t = \mathbb{E}^i_t\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\mathbf{1}_{\{\omega_{t+\Delta t}=\omega_3\}}\Big], \tag{15}$$

and the SPDs $\xi_{At}$ and $\xi_{Bt}$ are related by the following *change of measure equation*:

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = \frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\pi_A(\omega_{t+\Delta t})}{\pi_B(\omega_{t+\Delta t})}. \tag{16}$$

We find the SPDs from the first order conditions for consumptions in terms of investors' marginal utilities and Lagrange multipliers for capital constraints (10) by solving the investors' optimizations (7) via dynamic programming. To facilitate the tractability, using the existence of the SPD we first rewrite the budget equations (8)–(9) in a static form that expresses the current wealth in terms of current consumption and the expected discounted future wealth (e.g., Cox and Huang, 1989), and then solve the constrained optimization problem by employing the method of Lagrange multipliers. Lemma 1 below reports the equivalent optimization problem and the SPDs.

**Lemma 1 (Dynamic programming and the first order condition).**

1) *Let $V_i(W_{it}, v_t)$ denote the value function of investor $i$, where $v_t$ is an unspecified state variable. Then, the value function solves the following Hamilton-Jacobi-Bellman equation:*

$$V_i(W_{it}, v_t) = \max_{c_{it}} \left\{ u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i[V_i(W_{i,t+\Delta t}, v_{t+\Delta t})] \right\}, \tag{17}$$

*where the maximization is subject to a static budget and capital requirement constraints*

$$W_{it}+l_i D_t\Delta t = c_{it}\Delta t+\mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}W_{i,t+\Delta t}\right], \; W_{i,t+\Delta t} \geq k_i(S_{t+\Delta t}+(1-l_A-l_B)D_{t+\Delta t}\Delta t). \tag{18}$$

2) *The SPDs $\xi_{it}$ and optimal consumptions $c_{it}^*$ satisfy the following first order conditions:*

$$\frac{\xi_{i,t+\Delta t}}{\xi_{it}} = e^{-\rho\Delta t}\frac{(c_{i,t+\Delta t}^*)^{-\gamma_i} + \ell_{i,t+\Delta t}}{(c_{it}^*)^{-\gamma_i}}, \tag{19}$$

*where $\ell_{i,t+\Delta t} \geq 0$ is the Lagrange multiplier for capital constraint* (10) *satisfying the complementary slackness condition $\ell_{i,t+\Delta t}\Big(W_{i,t+\Delta t}^* - k_i\Big(S_{t+\Delta t} + (1-l_A-l_B)D_{t+\Delta t}\Delta t\Big)\Big) = 0.$*

We note, that the capital requirements of investors $A$ and $B$ never bind simultaneously because otherwise the market clearing conditions would be violated. Therefore, at each point of time, one of the investors' constraints does not bind, and hence, the unconstrained investor's marginal utilities can be used to derive the SPD using equation (19) with $\ell_{i,t+\Delta t} = 0$. We use the latter insight to derive $\xi_{i,t+\Delta t}/\xi_{it}$ below.

We conjecture and later verify that the equilibrium can be derived in terms of state variable $v_t$ given by the log-ratio of marginal utilities of investors evaluated at the investors' shares in the aggregate consumption $c_{it}^*/D_t$:

$$v_t = \ln\left(\frac{(c_{At}^*/D_t)^{-\gamma_A}}{(c_{Bt}^*/D_t)^{-\gamma_B}}\right). \tag{20}$$

Related literature on economies with heterogeneous investors typically derives equilibrium in terms of a consumption share of one of the investors in the aggregate consumption (e.g., Gârleanu and Pedersen, 2012; Chabakauri, 2013, 2015; Gârleanu and Panageas, 2014, among others). Using the consumption clearing condition in (11), we rewrite equation (20) in terms of consumption shares of investors $A$ and $B$, defined as $s = c^*_{At}/D_t$ and $1 - s = c^*_{Bt}/D_t$, and observe that variable $v_t$ is a decreasing function of share $s_t$, given by

$$v_t = \gamma_B \ln(1 - s_t) - \gamma_A \ln(s_t), \tag{21}$$

and hence, there is a one-to-one mapping between these two variables.

Our first contribution is to derive closed-form dynamics for the state variable $v_t$. We start with the case where constraints do not bind, and hence, Lagrange multipliers $\ell_i$ are zero. Next, using equation (20), we derive increment $v_{t+\Delta t} - v_t$ in terms of consumptions $c^*_{it}$, and then express this increment in terms of SPDs using the first order condition (19) with $\ell_i = 0$. Finally, using equation (16) relating the SPDs under the probability measures of investors $A$ and $B$, we obtain:[3]

$$v_{t+\Delta t} - v_t = \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A - \gamma_B}\right), \tag{22}$$

where $\pi_i(\omega_{t+\Delta t})$ are investor $i$'s probabilities of next-period state $\omega_{t+\Delta t} \in \{\omega_1, \omega_2, \omega_3\}$. Equation (22) gives the values of $v_{t+\Delta t}$ in next-period states $\omega_{t+\Delta t}$ in closed form.

Next, we look at the case of binding constraints. By $\overline{v}$ and $\underline{v}$ we denote the values of the state variable $v_t$ when constraints (10) of investors $A$ and $B$ bind, respectively, that is, $W_{i,t+\Delta t} = k_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t)$. Dividing both sides of the latter equation by $D_{t+\Delta t}$, we rewrite it in terms of wealth-aggregate consumption and pice-dividend ratios:

$$\Phi_A(\overline{v}) = k_A(1 - l_A - l_B)\Big(\Psi(\overline{v}) + \Delta t\Big), \quad \Phi_B(\underline{v}) = k_B(1 - l_A - l_B)\Big(\Psi(\underline{v}) + \Delta t\Big). \tag{23}$$

We note that $\underline{v} < \overline{v}$ because the constraint of investor $A$ binds when $A$'s consumption share $s_{t+\Delta t} = c_{A,t+\Delta t}/D_{t+\Delta t}$ is low and vice versa for investor $B$, and because low consumption share corresponds to larger state variable $v_t$, as discussed above.

Furthermore, we observe that $\underline{v}$ and $\overline{v}$ are reflecting boundaries for the state variable process $v_t$, and hence $\underline{v} \le v_t \le \overline{v}$. Intuitively, the capital constraint (10) restricts the investors' losses in such a way that their wealths, and hence also the consumption shares, never drop below a certain level. More formally, when investor $A$'s

[3] From equation (20) and the first order condition (19), we obtain:

$$v_{t+\Delta t} - v_t = \ln\left(\frac{(c^*_{A,t+\Delta t}/c^*_{At})^{-\gamma_A}}{(c^*_{B,t+\Delta t}/c^*_{Bt})^{-\gamma_B}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) = \ln\left(\frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right).$$

Using equation (16) for the SPDs, from the above equation we obtain equation (22).

constraint is binding so that $v_{t+\Delta t} = \overline{v}$, using the fact that Lagrange multiplier $\ell_{A,t+\Delta t}$ is positive and proceeding as in the derivation of equation (22), we find that $\overline{v} - v_t \leq \ln\Big(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\Big(D_{t+\Delta t}/D_t\Big)^{\gamma_A-\gamma_B}\Big)$. Similarly, for $v_{t+\Delta t} = \underline{v}$ we obtain that $\underline{v} - v_t \geq \ln\Big(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\Big(D_{t+\Delta t}/D_t\Big)^{\gamma_A-\gamma_B}\Big)$. The latter two inequalities and equation (22) for the dynamic of $v_t$ in the unconstrained case imply the following process for the state variable:

$$v_{t+\Delta t} = \max\left\{\underline{v}; \min\left\{\overline{v}; v_t + \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right)\right\}\right\}. \tag{24}$$

Process (24) can be further rewritten in a more tractable way in terms of discrete processes $w_t$ and $j_t$, as reported in Proposition 1 below.

**Proposition 1 (Closed-form state variable dynamics).**
*State variable $v_t$ follows dynamics:*

$$v_{t+\Delta t} = \max\Big\{\underline{v}; \min\{\overline{v}; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}\Big\}, \tag{25}$$

*where drift $\mu_v$, volatility $\sigma_v$, and the jump size $J_v$ are constants given by*

$$\mu_v = \frac{1}{2\Delta t}\left((\gamma_A - \gamma_B)\ln[(1+\mu_D\Delta t)^2 - \sigma_D^2\Delta t] + \ln\left(\frac{1-\lambda_B\Delta t}{1-\lambda\Delta t}\right)^2 + \ln(1-\delta^2\Delta t)\right), \tag{26}$$

$$\sigma_v = \frac{1}{2\sqrt{\Delta t}}\left((\gamma_A - \gamma_B)\ln\left(\frac{1+\mu_D\Delta t + \sigma_D\sqrt{\Delta t}}{1+\mu_D\Delta t - \sigma_D\sqrt{\Delta t}}\right) + \ln\left(\frac{1+\delta\sqrt{\Delta t}}{1-\delta\sqrt{\Delta t}}\right)\right), \tag{27}$$

$$J_v = (\gamma_A - \gamma_B)\ln(1+\mu_D\Delta t + J_D) + \ln\left(\frac{\lambda_B}{\lambda}\right) - \mu_v\Delta t. \tag{28}$$

Proposition 1 provides closed-form dynamics of the state variable $v_t$. The process for $v_t$ evolves as a discrete-time analogue of an arithmetic Brownian motion with a jump when constraints do not bind, and is reflected back into the unconstrained region when $v_t$ hits the boundary. Therefore, the capital constraint (10) naturally gives rise to a stationary distribution of the state variable, which we explore in the subsequent sections.

Our next step is to find the process for SPD and asset prices in the economy. When $v_t < \overline{v}$, and hence investor $A$'s constraint is not binding, the first order condition (19) implies that $\xi_{A,t+\Delta t}/\xi_{At} = e^{-\rho\Delta t}(c^*_{A,t+\Delta t})^{-\gamma_A}/(c^*_{A,t})^{-\gamma_A}$. When $v_t = \overline{v}$, investor $A$ is constrained but investor $B$ is unconstrained, and hence, the state price density can be obtained analogously in terms of investor $B$'s marginal utilities and then converted to correct beliefs of investor $A$ using the change of measure equation (16). After obtaining the SPD, we use equation

(14) for stock prices and equation (18) for investors' wealths to derive price-dividend and wealth-consumption ratios. Proposition 2 reports the results, and its proof in the Appendix provides further details of the derivation.

**Proposition 2 (Characterization of equilibrium in discrete time).**

1) *The state price density under the beliefs of investor A is given by:*

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\exp\Big(\max\{0; v_t+\mu_v\Delta t+\sigma_v\Delta w_t+J_v\Delta j_t-\overline{v}\}\Big), \quad (29)$$

*where investor A's time-t consumption share $s(v_t)$ solves equation* (21).

2) *The price-dividend ratio $\Psi$ and wealth-aggregate consumption ratios $\Phi_i$ are functions of the state variable $v$, and satisfy equations:*

$$\Psi(v_t) = \mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{D_{t+\Delta t}}{D_t}\Big(\Psi(v_{t+\Delta t})+\Delta t\Big)\right], \quad (30)$$

$$\Phi_i(v_t) = \mathbb{E}_t^A\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{D_{t+\Delta t}}{D_t}\Phi_i(v_{t+\Delta t})\right] + \Big(\mathbf{1}_{\{i=A\}}s(v_t)+\mathbf{1}_{\{i=B\}}(1-s(v_t))-l_i\Big)\Delta t, \quad (31)$$

*The stock price is then given by $S_t = (1-l_A-l_B)D_t\Psi_t$, and bounds $\underline{v}$ and $\overline{v}$ solve equations* (23). *Moreover, the matrix of asset payoffs is invertible if and only if $\sigma_t \neq 0$, where $\sigma_t$ is the stock return volatility in normal times.*

3) *Consider the constrained and unconstrained economies with the same value of state variable $v_t$. Then, the price-dividend ratio in the constrained economy is higher than in the unconstrained economy.*

## 3.1. Closed-form solution in a continuous-time limit

In this Section, we consider a continuous-time limit of the economy and provide closed-form expressions for price-dividend and wealth-consumption ratios, interest rates and risk premia. Taking limit $\Delta t \to 0$ allows rewriting equations (30) and (31) for the price-dividend and wealth-consumption ratios as differential-difference equations. For tractability, we derive ratios $\Psi_t$ and $\Phi_t$ in terms of a transformed ratio $\widehat{\Psi}(v;\theta)$, which satisfies a simpler equation reported in Lemma 2 below.

**Lemma 2 (Differential-difference equation).** *In the limit $\Delta t \to 0$, the price-dividend ratio $\Psi$ and wealth-aggregate consumption ratios $\Phi_i$ are given by:*

$$\Psi(v) = \widehat{\Psi}(v;-\gamma_A)s(v)^{\gamma_A}, \quad (32)$$

$$\Phi_i(v) = \Big((\mathbf{1}_{\{i=A\}}-\mathbf{1}_{\{i=B\}})\widehat{\Psi}(v;1-\gamma_A)+(\mathbf{1}_{\{i=B\}}-l_i)\widehat{\Psi}(v;-\gamma_A)\Big)\,s(v)^{\gamma_A}, \quad (33)$$

*where* $s(v)$ *solves equation* (21) *and* $\widehat{\Psi}(v;\theta)$ *satisfies a differential-difference equation*

$$\begin{aligned}
&\frac{\widehat{\sigma}_v^2}{2}\widehat{\Psi}''(v;\theta) + \Big(\widehat{\mu}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v\Big)\widehat{\Psi}'(v;\theta) \\
&\qquad - \left(\lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\widehat{\Psi}(v;\theta) \\
&\qquad + \lambda(1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v}; v+\widehat{J}_v\};\theta\Big) + s(v)^\theta = 0,
\end{aligned} \tag{34}$$

*subject to the reflecting boundary conditions*

$$\widehat{\Psi}'(\underline{v};\theta) = 0, \quad \widehat{\Psi}(\overline{v};\theta) - \widehat{\Psi}'(\overline{v};\theta) = 0, \tag{35}$$

*where constants* $\widehat{\mu}_v$*,* $\widehat{\sigma}_v$*, and* $\widehat{J}_v$ *are continuous-time limits of drift, volatility, and jump in equations* (26)–(28)*, and are given by:*

$$\widehat{\mu}_v = (\gamma_A - \gamma_B)\left(\mu_D - \frac{\sigma_D^2}{2}\right) + \lambda - \lambda_B - \frac{\delta^2}{2}, \tag{36}$$

$$\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta, \tag{37}$$

$$\widehat{J}_v = (\gamma_A - \gamma_B)\ln(1+J_D) + \ln\left(\frac{\lambda_B}{\lambda}\right). \tag{38}$$

We observe that equation (34) is linear, in contrast to economies with constraints directly imposed on trading strategies of investors (e.g., Gârleanu and Pedersen, 2012; Chabakauri, 2013, 2015; Rytchkov, 2014). Because investor $B$ is less risk averse and more optimistic, and also the aggregate output jump size $J_D$ is negative, the jump size of the state variable $\widehat{J}_v$, given by (38), is negative. Therefore, equation (34) is a differential-difference equation with a "delayed" argument in the fourth term on the left-hand side of the equation. This term is further complicated by the fact that the delayed argument is restricted to stay above the lower reflecting boundary $\underline{v}$, which gives rise to the dependence of the fourth term on a peculiar argument $\max\{\underline{v}; v+\widehat{J}_v\}$. Intuitively, the latter argument captures the impact of investors' decisions in anticipation of hitting their wealth constraint. Due to its linearity, the boundary value problem (34)–(35) can be easily solved numerically. However, despite an unusual delay argument, problem (34)–(35) has a unique closed-form solution, which we report in Proposition 3 below.

**Proposition 3 (Closed-form solutions and the existence of equilibrium).**

1) *In the limit* $\Delta t \to 0$ *the price-dividend ratio* $\Psi$ *and wealth-consumption ratios* $\Phi_i$ *are*

*given by equations* (32) *and* (33), *where function* $\widehat{\Psi}(v;\theta)$ *with parameter* $\theta$ *is given by:*

$$\widehat{\Psi}(v;\theta) = \int_{\underline{v}}^{v} s(y)^{\theta}\widehat{\psi}(v-y)dy + \frac{\int_{\underline{v}}^{\overline{v}} s(y)^{\theta}\left[\widehat{\psi}'(\overline{v}-y) - \widehat{\psi}(\overline{v}-y)\right]dy}{1 + H\left(\widehat{\psi}(\overline{v}-\underline{v}) - \int_0^{\overline{v}-\underline{v}}\widehat{\psi}(y)dy\right)}\left(1 - H\int_0^{v-\underline{v}}\widehat{\psi}(y)dy\right), \tag{39}$$

*where* $s(y)$ *solves equation* (21), and $\widehat{\psi}(x)$, $H$ *and some auxiliary variables are given by:*

$$\widehat{\psi}(x) = \frac{2}{\widehat{\sigma}_v^2}\sum_{n=0}^{\infty}\left[\left(\frac{2\lambda(1+J_D)^{1-\gamma_A}}{\widehat{\sigma}_v^2}\right)^n \frac{\exp\left((\zeta_+ + \zeta_-)(x + n\widehat{J}_v)/2\right)}{(\zeta_+ - \zeta_-)^{2n+1}n!}\right. \tag{40}$$

$$\left.\times\, Q_n\left(\frac{(\zeta_+ - \zeta_-)(x + n\widehat{J}_v)}{2}\right)\mathbf{1}_{\{x+n\widehat{J}_v\geq 0\}}\right], \tag{41}$$

$$Q_n(x) = \exp(-x)\sum_{m=0}^{n}(2x)^{n-m}\frac{(n+m)!}{m!(n-m)!} - \exp(x)\sum_{m=0}^{n}(-2x)^{n-m}\frac{(n+m)!}{m!(n-m)!}, \tag{42}$$

$$H = \lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2 - \lambda(1+J_D)^{1-\gamma_A}, \tag{43}$$

$$\zeta_{\pm} = -\frac{\widehat{\mu}_v + (1-\gamma_A)\widehat{\sigma}_v\sigma_D \mp \sqrt{(\widehat{\mu}_v + (1-\gamma_A)\widehat{\sigma}_v\sigma_D)^2 + 2\widehat{\sigma}_v^2\left(\lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)}}{\widehat{\sigma}_v^2}. \tag{44}$$

2) *Stock return volatility in normal times and the jump size* $J_t$ *are given by:*

$$\sigma_t = \sigma_D + \left(\frac{\widehat{\Psi}'(v_t;-\gamma_A)}{\widehat{\Psi}(v_t;-\gamma_A)} - \frac{\gamma_A(1-s(v_t))}{\gamma_A(1-s(v_t)) + \gamma_B s(v_t)}\right)\widehat{\sigma}_v, \tag{45}$$

$$J_t = \frac{(1+J_D)\widehat{\Psi}\left(\max\{\underline{v}; v_t + \widehat{J}_v\}; -\gamma_A\right)s\left(\max\{\underline{v}; v_t + \widehat{J}_v\}\right)^{\gamma_A}}{\widehat{\Psi}(v_t;-\gamma_A)s(v_t)^{\gamma_A}} - 1. \tag{46}$$

*The number of shares* $n^*_{i,St}$ *and leverage* $L_{it} = -b_{it}B_{it}$ *to market price* $S_t$ *ratio are given by:*

$$n^*_{i,St} = \frac{\Phi_i(v_t)\sigma_D + \Phi_i'(v_t)\widehat{\sigma}_v}{\Psi(v_t)\sigma_t}, \quad \frac{L_{it}}{S_t} = n_{i,St} - \frac{\Phi_i(v_t)}{\Psi(v_t)(1-l_A-l_B)}. \tag{47}$$

3) *Let* $\Psi_i$ *denote the price-dividend ratio in the economy populated only by investor* $i = A, B$. *If ratios* $\Psi_i$, *given by equations* (A58)-(A59) *in the Appendix, are positive and finite and* $\sigma_t \neq 0$, *then there exists an equilibrium such that investors' value functions are bounded and there exist reflecting boundaries* $\underline{v}$ *and* $\overline{v}$ *that satisfy equations* (23).

Although expression (39) depends on consumption share $s(y)$ implicitly defined by equation (21), we observe that the change of variable $x = s(y)$ allows rewriting (39) without implicit functions and fully in closed form.[4] However, because $s(y)$ is intuitive and easy to compute, we keep $s(v_t)$ in the expression for $\widehat{\Psi}(v;\theta)$. We note that, despite infinite summation, expression (41) for function $\widehat{\psi}(x)$ has only finite number of terms for a fixed $x$ because $\widehat{J}_v < 0$, and hence, indicators $\mathbf{1}_{\{x+n\widehat{J}_v \geq 0\}}$ become zero for sufficiently large $n$. The main advantage of having a closed-form solution is that it proves the existence and uniqueness of the solution satisfying equation (34) and boundary conditions (35). Nevertheless, the computations using the explicit formula remain tedious and the finite-difference method for solving boundary value problem (34)-(35) appears to be faster.[5]

We call the interval $v \in [\underline{v}, \underline{v} - \widehat{J}_v]$ in the state-space a period of *anxious economy*, similar to Fostel and Geanakoplos (2008).[6] When the economy falls into this state, even a small possibility of a crisis renders the capital requirement constraint binding and leads to deleveraging in the economy. To explore the economic effects of the anxious economy and to complete the characterization of equilibrium, we provide closed-form expressions for the interest rates $r_t$ and risk premia in normal times $\mu_t - r_t$, which can be easily obtained using previously derived equations for asset prices and the state price density. Proposition 4 below reports the results.

**Proposition 4 (Interest rates and risk premia in the limit).** *For a sufficiently small time-interval $\Delta t$ the interest rate $r_t$ and the risk premium $\mu_t - r_t$ in normal times have the following expansions:*

$$r_t = \begin{cases} \lambda + \rho + \gamma_A \mu_D - \dfrac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2 + \left(\dfrac{\gamma_A \sigma_D \widehat{\sigma}_v - \widehat{\mu}_v}{\gamma_B}\right)(1-s_t)\Gamma_t \\ \quad - \widehat{\sigma}_v^2 \left(\dfrac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2 + \dfrac{1}{2\gamma_A^2\gamma_B^2} s_t(1-s_t)\Gamma_t^3\right) \\ \quad - \lambda(1+J_D)^{-\gamma_A}\left(\dfrac{s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)}{s_t}\right)^{-\gamma_A} + O(\Delta t), \ \text{for } \underline{v} < v_t < \overline{v}, \\ \dfrac{(1-s_t)\Gamma_t\Big(\mathbf{1}_{\{v=\underline{v}\}} - \mathbf{1}_{\{v=\overline{v}\}}\Big) - \gamma_B}{2\gamma_B\sqrt{\Delta t}}\widehat{\sigma}_v + O(1), \ \text{for } v = \underline{v} \text{ or } v = \overline{v}. \end{cases} \tag{48}$$

[4]Despite the fact that $s(y)$ is not available in closed form from equation (21) we observe that its inverse function is given by $s^{-1}(x) = \gamma_B \ln(x) - \gamma_A \ln(1-x)$. Therefore, the change of variable $x = s(y)$ can be performed in closed form, similar to Chabakauri (2015).

[5]In the absence of crises (i.e., $\lambda = \lambda_B = 0$), in the economy where $\gamma_B = \gamma_A$ and risk aversions are integers, the price-dividend ratio $\Psi(v)$ is available in closed form as a finite sum of elementary functions. For general risk aversions, the price-dividend ratio can be obtained in terms of hypergeometric functions.

[6]However, in contrast to Fostel and Geanakoplos (2008), in our economy the disagreement about the consumption growth dynamics does not increase during these periods.

$$
\begin{aligned}
\mu_t - r_t = &\left(\gamma_A \sigma_D - \frac{(1-s_t)\Gamma_t \widehat{\sigma}_v}{\gamma_B} + \frac{(1-s_t)\Gamma_t \widehat{\sigma}_v (\mathbf{1}_{\{v=\underline{v}\}} + \mathbf{1}_{\{v=\overline{v}\}}) - \gamma_B \widehat{\sigma}_v \mathbf{1}_{\{v=\overline{v}\}}}{2\gamma_B}\right)\sigma_t \\
&- \lambda(1+J_D)^{-\gamma_A} J_t \left(\frac{s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)}{s_t}\right)^{-\gamma_A} + O(\sqrt{\Delta t}),
\end{aligned} \tag{49}
$$

*where the continuous-time drift* $\widehat{\mu}_v$*, volatility* $\widehat{\sigma}_v$*, and* $\widehat{J}_v$ *of the state variable* $v$ *are given by equations* (36)–(38)*, volatility* $\sigma_t$ *and jump size* $J_t$ *are given by equations* (45)–(46)*, respectively, and* $\Gamma_t \equiv \gamma_A\gamma_B / \Big(\gamma_A(1-s_t) + \gamma_B s_t\Big)$ *is the risk aversion of a representative investor in the economy.*

Proposition 4 provides tractable expressions for the interest rates and risk premia. The effects of capital requirements on interest rates and risk premia arise due to the investors' concern that when the economy is close to the boundary $\underline{v}$ a potential crisis may render the constraints binding next period. As a result, the investors close to default invest more in bonds, which leads to lower interest rates and higher risk premia to provide compensation for holding risky assets. In particular, the last term in the first equation in (48) for the interest rate quantifies the impact of capital requirements on precautionary savings due to a downward jump in the aggregate consumption, which we further discuss in Section 4.

Equations (48) and (49) also contain terms with indicator functions $\mathbf{1}_{\{v=\underline{v}\}}$ and $\mathbf{1}_{\{v=\overline{v}\}}$, which are non-zero only at the boundaries $\underline{v}$ and $\overline{v}$. For the interest rate $r_t$ these terms have the order of magnitude proportional to $1/\sqrt{\Delta t}$, and hence, the interest rate has singularities at the boundaries $\underline{v}$ and $\overline{v}$ when $\Delta t \to 0$. Similar singularities at the boundaries arise in a continuous-time model of Detemple and Serrat (2003). Our discrete-time analysis sheds further light on these singularities by uncovering their order of magnitude when time is discrete. In particular, our analysis reveals for the first time that although the annualized interest rate $r_t$ becomes $-\infty$ when $\Delta t \to 0$, the per-period rate $r_t\Delta t$ exists, is finite, and has an order of magnitude $O(\sqrt{\Delta t})$. Therefore, the singularity does not pose any difficulty in a discrete-time economy or its continuous-time limit.

The intuition for the singularity is that in the interior region $\underline{v} < v_t < \overline{v}$ only a downward jump poses the risk of making time-$(t+\Delta t)$ wealth negative (assuming that $\Delta t$ is sufficiently small) whereas at the boundary $v_t = \underline{v}$ even a small negative shock $\Delta w_t = -\sqrt{\Delta t}$ may lead to a default. Consequently, when the capital requirement of an investor binds at time $t$, the investor spends time-$t$ labor income mainly on consumption and risk-free bonds. Therefore, the interest rate sharply decreases and Sharpe ratio increases to provide fair compensation for holding risky assets.

## 3.2. Stationary distribution of consumption share

Due to the presence of the reflecting boundaries, the process for the state variable $v$ is stationary and $v_t \in [\underline{v}, \overline{v}]$ at all times. Similarly, the consumption share $s$ is bounded and stationary. Intuitively, the stationarity arises because the capital requirements (10) protect investors against losing completely their shares of aggregate consumption. Absent any frictions, state variable $v$ follows an arithmetic Brownian motion with a jump, and hence, the equilibrium is non-stationary. Therefore, labor incomes alone do not guarantee stationarity, as we further explain in Section 4. We compute the transition densities and the stationary probability density function (PDF) of consumption share $s$ in closed form in the continuous-time limit for the case when there is no risk of a crisis in the economy, that is, $J_D = 0$ and $\lambda = \lambda_B = 0$. Proposition 5 below reports the result.

**Proposition 5 (Stationary distribution of consumption share).** *Suppose, $\lambda = \lambda_B = 0$. Then, the PDF $f(s, \tau; s_t; \tau)$ of consumption share $s$ at time $\tau$ conditional on observing share $s_t$ at time $t$ is given in closed form by expression* (A103) *in the Appendix. Furthermore, the stationary PDF of consumption share $s$ is given by:*

$$f(s) = \frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(\frac{\gamma_A}{s} + \frac{\gamma_B}{1-s}\right)\frac{\left((1-s)^{\gamma_B}/s^{\gamma_A}\right)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}}{\left((1-\underline{s})^{\gamma_B}/\underline{s}^{\gamma_A}\right)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2} - \left((1-\overline{s})^{\gamma_B}/\overline{s}^{\gamma_A}\right)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}}\mathbf{1}_{\{\underline{s}\leq s\leq \overline{s}\}}, \quad (50)$$

*where $\widehat{\mu}_v = (\gamma_A - \gamma_B)(\mu_D - \sigma_D^2/2) - \delta^2/2$, $\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta$, $\mathbf{1}_{\{\underline{s}\leq s\leq \overline{s}\}}$ is an indicator function and $\underline{s}$ and $\overline{s}$ are the bounds on the consumption share $s$, which solve equation* (23) *for $\overline{v}$ and $\underline{v}$, respectively.*

One important economic implication of the stationarity of equilibrium is that in our constrained economy the investors with irrational beliefs survive in the long-run. This finding is in contrast to frictionless economies where irrational investors lose wealth gradually over time, as first conjectured by Friedman (1953) and theoretically verified in various economic settings (e.g., Blume and Easley, 2001; Yan, 2008; Chabakauri, 2015, among others).[7] Stationary PDF (50) has a particularly simple form when the investors have identical risk aversions, as reported in Corollary 1 below. Figure 2 plots the stationary PDF (51) and transition densities $f(s, t; s_0, 0)$, given by equation (A103) in the Appendix A, for the case of heterogeneous risk aversions $\gamma_A = 2$ and $\gamma_B = 1.5$ and beliefs, different horizons $t$, starting point $s_0 = 0.2$, and boundaries $\underline{s} = 0.1$ and $\overline{s} = 0.9$. The stationary

[7] Borovička (2015) demonstrates the stationary of equilibrium in economies with Epstein-Zin investors disagreeing about the output growth rate under certain restrictions on the risk aversions and intertemporal elasticities of substitution of investors. The stationarity in our model arises due to the capital requirement constraints and does not depend on investors' risk aversions.

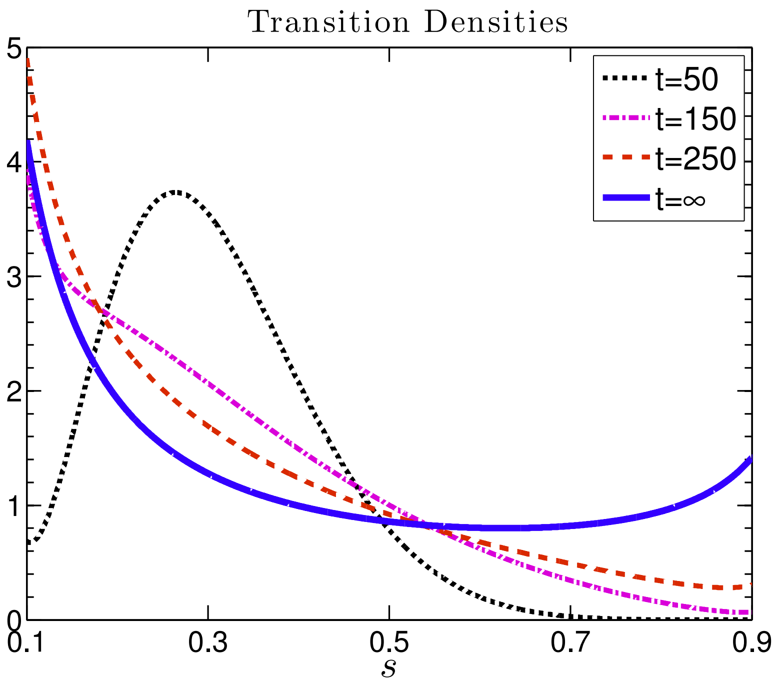


**Figure 2**
**Convergence to stationary distribution**
The Figure shows transition densities $f(s,t;s_0,0)$ for the starting point $s_0 = 0.2$ and the stationary distribution $f(s)$ (i.e., density for $t = \infty$) for the economy with $\gamma_A = 2$, $\gamma_B = 1.5$, $\mu_D = 0.018$, $\sigma_D = 0.032$, $\lambda = \lambda_B = 0$, $\rho = 0.02$, $\delta = 0.1125$, $\underline{s} = 0.1$, $\overline{s} = 0.9$.

distribution on Figure 2 is a non-monotone function of $s$, and hence, both rational and irrational investors can occasionally have large consumption shares in equilibrium.

**Corollary 1 (Stationary distribution of $s$ for identical risk aversions).** *Suppose, investors have identical risk aversions $\gamma_A = \gamma_B = \gamma$. Then, the stationary distribution of consumption share $s$ is given by*

$$f(s) = \frac{\gamma}{s(1-s)} \frac{\big(s/(1-s)\big)^{\gamma}}{\big(\overline{s}/(1-\overline{s})\big)^{\gamma} - \big(\underline{s}/(1-\underline{s})\big)^{\gamma}} \mathbf{1}_{\{\underline{s} \le s \le \overline{s}\}}. \tag{51}$$

Corollary 1 reveals a surprising result that the PDF of consumption share $s$ does not directly depend on the disagreement parameter $\delta$, the aggregate consumption mean growth $\mu_D$ and volatility $\sigma_D$ when investors have identical risk aversions. These parameters only affect the stationary PDF (51) via the reflecting boundaries $\underline{s}$ and $\overline{s}$.

# 4. Analysis of Equilibrium

In this section, we provide the analysis of equilibrium. We assume the following parameters of the aggregate consumption process $\mu_D = 1.8\%$, $\sigma_D = 3.2\%$, $J_D = -20\%$, and the crisis intensities of investors $A$ and $B$ are $\lambda = 0.017$ and $\lambda_B = 0.01$, respectively. Furthermore, we assume that the disagreement parameter is $\delta = 0.1125$, which corresponds to the mean growth rate (6) under investor $B$'s probabilities equal to $1.2\mu_D$, that is, 20% higher than the true rate $\mu_D$. For convenience, we here consider and plot the equilibrium processes as functions of consumption share $s_t = c^*_{At}/D_t$ because $s$ lies in the interval $[0, 1]$. We note that consumption share $s$ is countercyclical in the sense that it tends to increase (decrease) following negative (positive) shocks to aggregate output growth rate $dD_t/D_t$. The countercyclicality of $s$ is due to the fact that the aggregate wealth and consumption shift to pessimist $A$ (optimist $B$) following negative (positive) shocks to output. Hence, we will say that a process is procyclical (countercyclical) if that process is a decreasing (increasing) function of $s$. We assume that constraint tightness parameters in (10) are given by $k_A = k_B = 0$, so that asset positions are required to be fully collateralized, and set lower and upper reflecting bounds of consumption shares to $\underline{s} = 0.1$ and $\overline{s} = 0.9$.[8]

Figure 3 depicts investor $B$'s leverage/market ratio $L_t/S_t$ and stock holdings $n_{Bt}$ in the constrained (solid line) and unconstrained (dashed line) economies for three cases with different risk aversions: $\gamma_A = \gamma_B = 0.9$ (panels (a.i)–(b.i)), $\gamma_A = \gamma_B = 2$ (panels (a.ii)–(b.ii)), $\gamma_A = 2, \gamma_B = 1.5$ (panels (a.iii)–(b.iii)).[9] Panels (a.i)–(a.iii) demonstrate the cyclicality of leverage in our economy. The leverage is lowest when either investor $A$ or investor $B$ bind on their constraints. Intuitively, when $s = \overline{s}$, investor $B$'s financial wealth is zero, and hence, $B$ cannot borrow because otherwise there might be a possibility of financial wealth becoming negative if the economy is hit by adverse shocks. When $s = \underline{s}$, investor $A$'s financial wealth is zero and the labor income $l_A D_t \Delta t$ is infinitesimally small in the continuous-time limit. Therefore, investor $A$ cannot supply credit, and hence the liquidity "dries up" for a certain period until investor $A$ accumulates sufficient savings.

Panels (b.i)–(b.iii) demonstrate that the cyclicality of leverage naturally induces the cyclicality in trading strategies. In particular, higher leverage allows investor $B$ to acquire

[8]Instead of finding bounds $\underline{s}$ and $\overline{s}$ by first solving equation (23) and then equation (21), we pick them directly. The implied labor income shares $l_i$ and the tightness parameters $k_i$ giving rise to these bounds can be backed out from equations (23), where $\underline{v}$ and $\overline{v}$ solve equation (20) for $\overline{s}$ and $\underline{s}$, respectively. We justify this approach by noting that exact values of income shares $l_i$ do not affect the qualitative results.

[9]Following Longstaff and Wang (2012), we define the leverage/market ratio as the ratio of total debt to the stock market value in the economy.

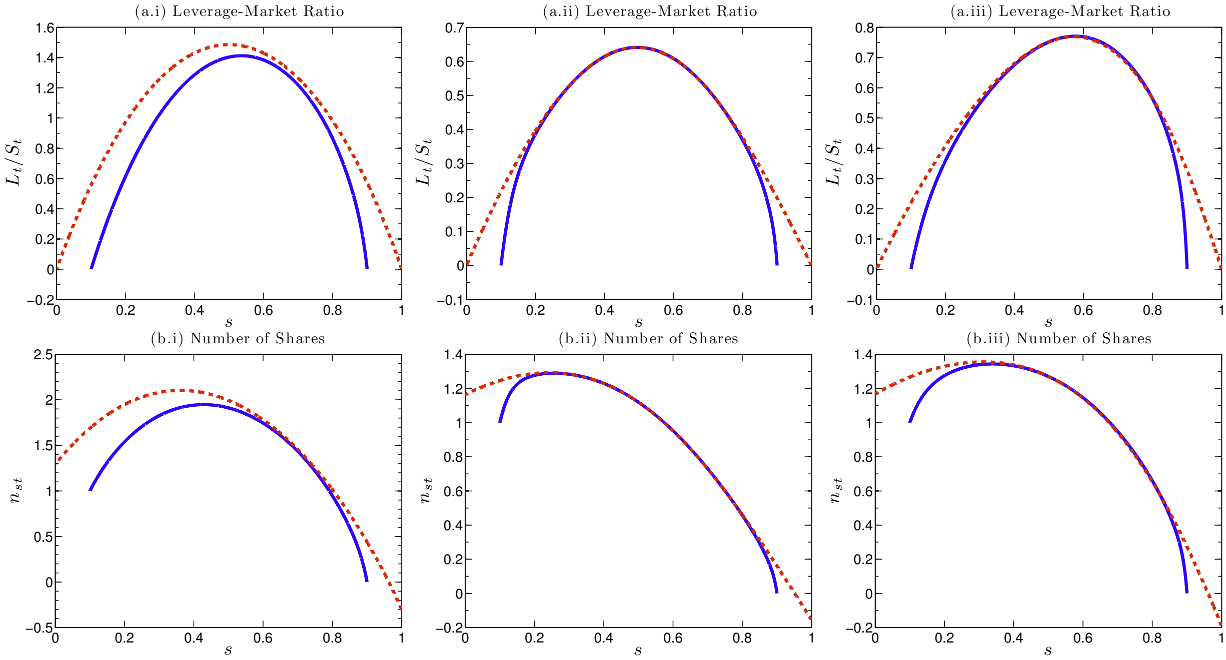


**Figure 3**
**Leverage and stock holdings of optimistic and less risk averse investor $B$**
The Figure depicts optimistic and less risk averse investor $B$'s leverage/market price ratio $L_t/S_t$ and number of shares $n_{st}$ as functions of consumption share $s_t = c^*_{At}/D_t$. The solid and dashed lines correspond to constrained and unconstrained economies, respectively. We set $\mu_D = 0.018$, $\sigma_D = 0.032$, $J_D = -0.2$, $\lambda = 0.017$, $\lambda_B = 0.01$, $\rho = 0.02$, $\delta = 0.1125$, $k_A = k_B = 0$, $\underline{s} = 0.1$, $\overline{s} = 0.9$ and 1) $\gamma_A = \gamma_B = 0.9$ (panels (a.i)-(b.i)), 2) $\gamma_A = \gamma_B = 2$ (panels (a.ii)-(b.ii)), 3) $\gamma_A = 2, \gamma_B = 1.5$ (panels (a.iii)-(b.iii)).

more shares. From panels (b.i)–(b.iii) we also observe that in the unconstrained economy investor $B$ shorts stocks when consumption share $s$ is close to its upper bound $\overline{s}$, despite being more optimistic and less risk averse than investor $A$. The intuition is that in bad times, following a sequence of negative shocks to output, investor $B$ shorts stocks to finance current consumption and backs short position by the pledgeable labor income. From the perspective of investor $B$, the stream of labor income $l_B D_t \Delta t$ is equivalent to dividend payments from holding $\hat{n}_B = l_B/(1 - l_A - l_B)$ units of non-tradable shares in the Lucas tree. Short-selling allows the investor to circumvent the inability to trade $\hat{n}_B$ shares and freely adjust the effective share in the tree given by $\hat{n}_B + n_{B,st}$. The effective share $\hat{n}_B + n_{B,st}$ and consumption share $1 - s$ of investor $B$ may decline to zero following a sequence of bad shocks, and the financial wealth $W_{Bt}$ may become negative. The trading strategy of investor $A$ can be analyzed similarly. Moreover, investor $A$ has an additional motive to short stocks because of being more pessimistic than investor $B$.

Short selling in the unconstrained economy mitigates the non-tradability of labor income, and hence, the equilibrium has similar properties as in the economies without labor income. In particular, the distribution of consumption share $s$ is non-stationary as in the latter economies (e.g., Chabakauri, 2015), and, in general, only one of the investors survives in the long run. In contrast to the unconstrained economy, in the constrained economy financial wealths of investors are required to stay positive. As a result, consumption share $s$ of investor $A$ has low $\underline{s} > 0$ and upper $\overline{s} < 1$ bounds, which induce stationarity.

Figure 4 depicts interest rates $r_t$, Sharpe ratios $(\mu_t - r_t)/\sigma_t$ in normal times, price-dividend ratios $\Psi$, excess stock return volatilities in normal times $(\sigma_t - \sigma_D)/\sigma_D$ in the constrained (solid line) and unconstrained (dashed line) economies for three cases with different risk aversions: $\gamma_A = \gamma_B = 0.9$ (panels (a.i)–(d.i)), $\gamma_A = \gamma_B = 2$ (panels (a.ii)–(d.ii)), $\gamma_A = 2, \gamma_B = 1.5$ (panels (a.iii)–(d.iii)). Our results indicate that the qualitative effect of the capital requirement constraint on equilibrium is similar for different risk aversions. Below, we discuss the results and provide the economic intuition.

Panels (a.i)–(a.iii) show the interest rates $r_t$ in the constrained and unconstrained economies.[10] We find that the interest rate declines sharply when the economy enters into an anxious state close to the boundary $\overline{s}$ in which even a small possibility of a crisis state next period makes the constraint of investor $B$ binding. The economic intuition is as follows. In the unconstrained economy a crisis around state $\overline{s}$ generates wealth transfer to the pessimistic and more risk averse investor $A$ and increases her consumption share $s$ beyond $\overline{s}$. However, in the constrained economy consumption share $s$ remains bounded by $\overline{s}$. Therefore, following a crisis, investor $A$'s marginal utility is higher in the constrained than in the unconstrained economy. As a result, investor $A$ is more willing to smooth consumption in the constrained economy and the interest rate declines due to the precautionary savings motive. In particular, the investor buys more bonds, which drives interest rates down. Panels (b.i)–(b.iii) show that the Sharpe ratio increases to compensate investor $A$ who is purchasing the risky assets from investor $B$. Our results on interest rates and Sharpe ratios demonstrate that the rare crises and capital requirement constraints reinforce the effects of each other. In particular, the decreases in interest rates and increases in Sharpe ratios during anxious times arise only when both the crises and the constraints (10) are present at the same time.

From panels (c.i)–(c.iii) we observe that the capital requirements give rise to higher price-dividend ratio $\Psi$ than in the unconstrained economy, $\Psi_t^{constr} - \Psi_t^{unc} > 0$, consistent

[10]We exclude the singularities in the dynamics of $r_t$ and focus on the dynamics in the unconstrained region because the economy spends an infinitesimal amount of time at the boundaries.

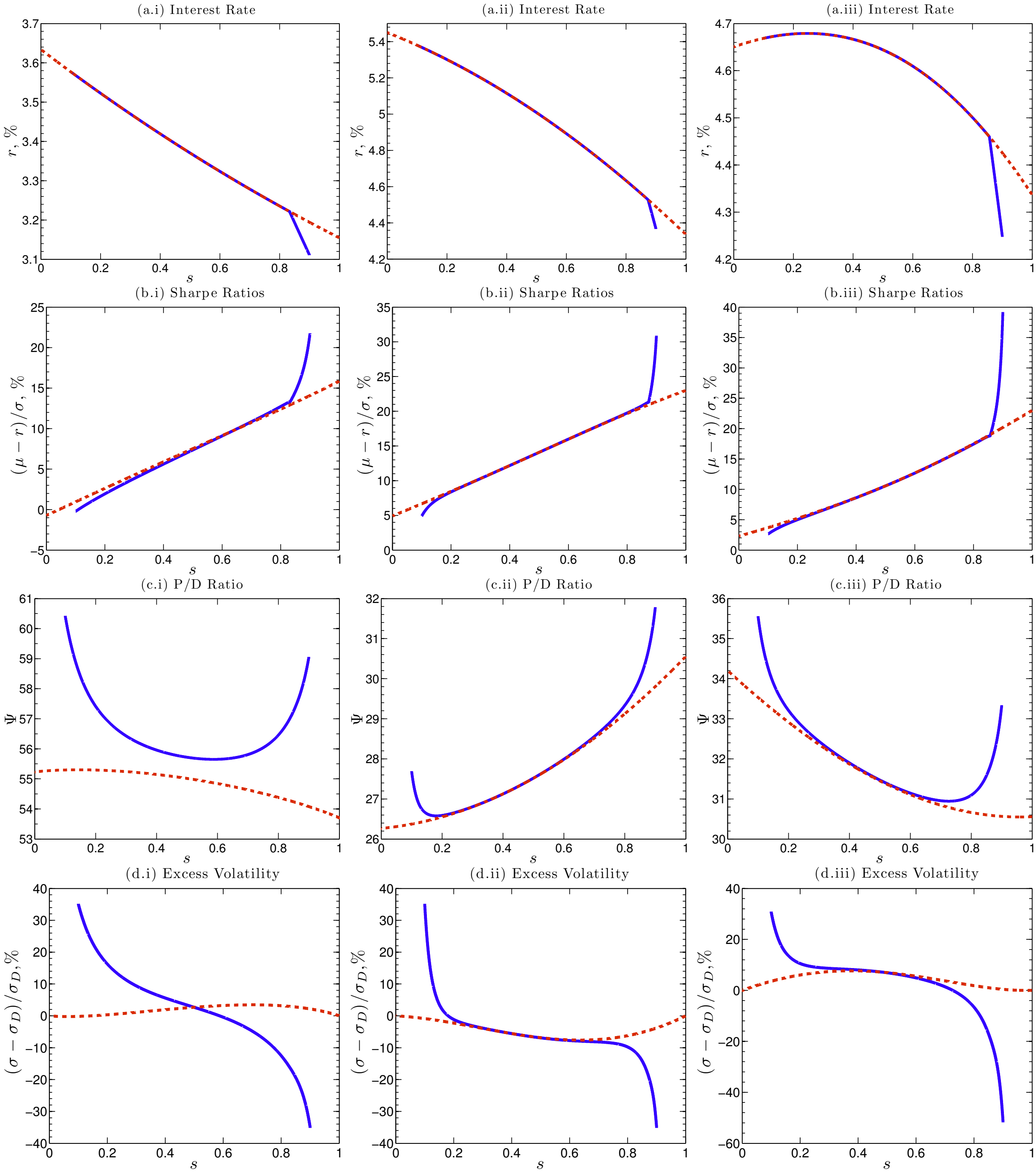


**Figure 4**
**Equilibrium processes**

The Figure depicts interest rates $r_t$, Sharpe ratios $(\mu_t - r_t)/\sigma_t$, price-dividend ratios $\Psi_t$ and excess volatilities $(\sigma_t - \sigma_D)/\sigma_D$ as functions of $s_t = c^*_{At}/D_t$. The solid and dashed lines correspond to constrained and unconstrained cases, respectively. We set $\mu_D = 0.018$, $\sigma_D = 0.032$, $J_D = -0.2$, $\lambda = 0.017$, $\lambda_B = 0.01$, $\rho = 0.02$, $\delta = 0.1125$, $k_A = k_B = 0$, $\underline{s} = 0.1$, $\overline{s} = 0.9$ and 1) $\gamma_A = \gamma_B = 0.9$ (panels (a.i)-(d.i)), 2) $\gamma_A = \gamma_B = 2$ (panels (a.ii)-(d.ii)), 3) $\gamma_A = 2, \gamma_B = 1.5$ (panels (a.iii)-(d.iii)).

with Proposition 2, and the increases in $\Psi$ are larger around the reflecting boundaries $\underline{s}$ and $\overline{s}$. As a result, the price-dividend ratios become U-shaped and sensitive to small shocks around times when constraints become binding. We provide the intuition below.

Consider the case when the economy evolves close to the boundary $\overline{s}$, where investor $B$'s constraint is likely to bind but investor $A$ is unconstrained. Because investor $A$'s constraint does not bind the state price density $\xi_{At}$ is proportional to investor $A$'s marginal utility of consumption $(c^*_{At})^{-\gamma_A}$. In the constrained economy the consumption share of investor $A$ is capped by $\overline{s} < 1$ whereas in the unconstrained economy it can increase beyond $\overline{s}$. Therefore, the marginal utility of investor $A$ is expected to be higher in the constrained economy than in the unconstrained one, and hence stocks are more valuable in the constrained economy around the boundary $\overline{s}$. Close to the boundary $\underline{s}$ investor $A$'s constraint is likely to bind whereas investor $B$ is unconstrained. Similarly to the above, the marginal utility of investor $B$ is higher in the constrained economy, which generates a spike in the price-dividend ratio around $\underline{s}$.

There are two additional economic forces that contribute to higher stock prices in the constrained economy. First, the constraint curbs short-selling by pessimist $A$, which contributes to higher stock prices (e.g., Harrison and Kreps, 1978). Second, the stock can be used as collateral against which optimistic investors can borrow in good times in lieu of future labor income, which makes the stock more valuable.

The dynamics of price-dividend ratios determines the dynamics of volatilities. The results on panels (d.i)–(d.iii) demonstrate that the constraint makes volatilities more procyclical relative to the unconstrained case. Moreover, the constraint reduces stock return volatility in bad times (around $\overline{s}$) and increases it in good times (around $\underline{s}$). This is because the price-dividend ratio becomes more procyclical in good times (i.e., around $\underline{s}$) and more countercyclical in bad times (i.e., around $\overline{s}$). Therefore, the stock becomes excessively volatile in good times because both the price-dividend ratio and the dividend move in the same direction, amplifying each other. Similarly, the volatility decreases in bad times because the price-dividend ratio and the dividend move in opposite directions and cancel the effects of each other. The decreases in volatilities in bad times are in line with the findings in the previous literature that constraining liquidity may lead to lower volatilities (e.g., Chabakauri, 2013, 2015; Brunnermeier and Sannikov, 2014, among others).[11]

The boundary conditions (35) allow us to explore volatility $\sigma_t$ near the reflecting bound-

[11] We also note that $\sigma_t > 0$ for all $s$, and hence, as shown in Proposition 3, the matrix of asset payoffs is invertible under our calibration of model parameters. This verifies our assumption in the beginning of Section 3 that $\sigma_t \neq 0$.

aries using closed form expressions reported in Corollary 2 below.

**Corollary 2 (Stock return volatility at the boundaries).** *Stock return volatility in normal times $\sigma_t$ satisfies the following boundary conditions:*

$$\sigma(\underline{s}) = \sigma_D + \frac{\underline{s}\gamma_B\widehat{\sigma}_v}{\gamma_A(1-\underline{s}) + \gamma_B\underline{s}} > \sigma_D, \tag{52}$$

$$\sigma(\overline{s}) = \sigma_D - \frac{(1-\overline{s})\gamma_A\widehat{\sigma}_v}{\gamma_A(1-\overline{s}) + \gamma_B\overline{s}} < \sigma_D, \tag{53}$$

*where $\underline{s}$ and $\overline{s}$ are consumption shares of investor A, which solve equation* (21) *for the boundary values of the state variables $\overline{v}$ and $\underline{v}$, respectively.*

By continuity of the volatility, the inequalities (52) and (53) also hold in a small vicinity of boundaries $\underline{s}$ and $\overline{s}$, respectively. The inequalities in Corollary 2 confirm our intuition and are consistent with panels (b.i)–(b.iii). In particular, the volatility is higher around $\underline{s}$ (good times) and lower around $\overline{s}$ (bad times). The empirical evidence suggests that the volatility tends to be higher in bad times (e.g., Schwert, 1989). However, the literature argues that high volatility in recessions can be explained by higher uncertainty about the economic growth and learning effects (e.g., Veronesi, 1999) which are absent in our model.

Our results can potentially explain volatility crashes, which arise endogenously in our economy. In particular, although volatility $\sigma_t$ is a continuous function of consumption share $s$, it becomes very steep close to the reflecting boundaries in the constrained economy, which in the data might be difficult to distinguish from a discontinuous jump. Furthermore, we note that for the cases when $\gamma_A > 1$ and $\gamma_B > 1$ on panels (d.ii) and (d.iii), the plot of volatility $\sigma_t$ can be subdivided into three parts with distinct dynamics. More specifically, the market is very volatile in a region around $\underline{v}$, calm around $\overline{v}$, and moderate in between. In the data such dynamics would lead to time-varying volatility clustering, consistent with the empirical evidence (e.g., Bollerslev, 1987).

# 5. Conclusion

In this paper, we develop a parsimonious and tractable theory of asset pricing under capital requirement constraints. We show that requiring investors to collateralize their trades has significant effects on asset prices and their moments. We find that constraints decrease interest rates and increase Sharpe ratios when optimistic investors are close to default boundaries. The constraints increase price-dividend ratios, amplify volatilities in good states and dampen them in bad states, and hence, capital requirements emerge as viable instruments for stabilizing markets in bad times. The tractability of our model allows us to obtain asset prices and the distributions of consumption shares in closed form.

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# Appendix A: Proofs

**Proof of Lemma 1.**

1) We start by demonstrating the equivalence of the dynamic (8)–(9) and static (18) budget constraints. Multiplying equation (9) by $\xi_{i,t+\Delta t}/\xi_{it}$, taking expectation operator $\mathbb{E}_t^i[\cdot]$ on both sides, and using equations (13)–(15) for asset prices, we obtain:

$$\mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}W_{i,t+\Delta t}\right] = b_{it}B_t + n_{it}(S_t, P_t)^\top. \tag{A1}$$

From the budget constraint equation (8), we observe that the right-hand side of (A1) equals $W_{it} + l_i D_t \Delta t$, and hence, we obtain the static budget constraint in (18). Conversely, if there exists $W_{i,t+\Delta t}$ satisfying constraints in (18) there exist trading strategies $b_{it}$ and $n_{it}$ that replicate $W_{i,t+\Delta t}$ because the underlying market is effectively complete (i.e., the payoff matrix is invertible). Then, rewriting the optimization problem (7) in a recursive form, we obtain the dynamic programming equation (17) for the value function.

2) Consider the following Lagrangian:

$$\begin{aligned}\mathscr{L} &= u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i\Big[V_i(W_{i,t+\Delta t}, v_{t+\Delta t})\Big] \\ &+ \eta_{it}\left(W_{it} + l_i D_t \Delta t - c_{it}\Delta t - \mathbb{E}_t^i\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}W_{i,t+\Delta t}\Big]\right) \\ &+ \mathbb{E}_t^i\Big[e^{-\rho\Delta t}\ell_{i,t+\Delta t}\Big(W_{i,t+\Delta t} - k_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t)\Big)\Big],\end{aligned} \tag{A2}$$

where multiplier $\ell_{i,t+\Delta t}$ satisfies the complementary slackness condition $\ell_{i,t+\Delta t}\Big(W_{i,t+\Delta t} - k_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t)\Big) = 0$. Differentiating the Lagrangian (A2) with respect to $c_{it}$ and $W_{i,t+\Delta t}$, we obtain:

$$u_i'(c_{it}^*) = \eta_{it}, \tag{A3}$$

$$e^{-\rho\Delta t}\left(\frac{\partial V_i(W_{t+\Delta t}, v_t)}{\partial W} + \ell_{i,t+\Delta t}\right) = \eta_{i,t}\frac{\xi_{i,t+\Delta t}}{\xi_{it}}. \tag{A4}$$

By the envelope theorem (e.g., Mas-Colell, Whinston and Green, 1995; Back, 2010):

$$\frac{\partial V_i(W_{i,t+\Delta t}, v_{t+\Delta t})}{\partial W} = u_i'(c_{i,t+\Delta t}^*). \tag{A5}$$

Substituting the partial derivative of the value function (A5) and the marginal utility (A3) into equation (A4), and then dividing both sides of the equation by $u_i'(c_{it}^*)$, we obtain the expression for the SPD (19). ■

**Proof of Proposition 1.** We look for coefficients $\mu_v$, $\sigma_v$ and $J_v$ such that when the constraints do not bind the increment $v_{t+\Delta t} - v_t$ is given by:

$$
\begin{aligned}
\mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\
&= \ln\left(\frac{\pi_B(w_{t+\Delta t})}{\pi_A(w_{t+\Delta t})}\right) + (\gamma_A - \gamma_B)\ln(1+\mu_D\Delta t + \sigma_D \Delta w_t + J_v \Delta j_t).
\end{aligned}
\tag{A6}
$$

Next, we write identity (A6) in each of the states $\omega_{t+\Delta t} \in \{\omega_1, \omega_2, \omega_3\}$ and obtain the following system of three linear equations with three unknowns $\mu_v$, $\sigma_v$ and $J_v$:

$$
\begin{aligned}
\mu_v \Delta t + \sigma_v \sqrt{\Delta t} &= \ln\left(\frac{(1-\lambda_B\Delta t)(1+\delta\Delta t)}{1-\lambda\Delta t}\right) + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t + \sigma_D\sqrt{\Delta t}), \\
\mu_v \Delta t - \sigma_v \sqrt{\Delta t} &= \ln\left(\frac{(1-\lambda_B\Delta t)(1-\delta\Delta t)}{1-\lambda\Delta t}\right) + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t - \sigma_D\sqrt{\Delta t}), \\
\mu_v \Delta t + J_v &= \ln\left(\frac{\lambda_B}{\lambda}\right) + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t + J_D).
\end{aligned}
\tag{A7}
$$

Solving the above system, we obtain $\mu_v$, $\sigma_v$ and $J_v$ reported in Proposition 1. ■

**Proof of Proposition 2.** 1) First, we derive the SPD $\xi_{At}$ under the correct beliefs of investor $A$. When investor $A$'s constraint does not bind, substituting $c^*_{At} = s(v_t)D_t$ into the first order condition (19) we find that

$$
\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}. \tag{A8}
$$

Equation (A8) is consistent with SPD (29) because when the constraint does not bind $v_{t+\Delta t} = v_t + \mu_v\Delta t + \sigma_v \Delta w_t + J_v \Delta j_t < \overline{v}$, and hence the exponential term in (29) vanishes.

When the constraint of investor $A$ is binding, the constraint of investor $B$ does not bind. This is because both constraints cannot bind simultaneously lest to violate the market clearing conditions. Therefore, the ratio $\xi_{B,t+\Delta t}/\xi_{Bt}$ is given by FOC (19) for investor $B$ with $\ell_B = 0$. Using equation (16), we rewrite the latter SPD under the correct beliefs of investor $A$ and obtain:

$$
\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{1-s(v_{t+\Delta t})}{1-s(v_t)}\right)^{-\gamma_B}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}. \tag{A9}
$$

Next, from equation (21) for consumption share $s$ we find that $(1-s_t)^{-\gamma_B} = e^{-v_t}s_t^{-\gamma_A}$. Substituting the latter equality into equation (A9), and also using equation (A6) for the increment $v_{t+\Delta t} - v_t$, we obtain:

$$\begin{aligned}\frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A} e^{v_t - v_{t+\Delta t}}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B} \\ &= e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\exp\{v_t - v_{t+\Delta t} + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}.\end{aligned} \tag{A10}$$

The fact that the constraint of investor $A$ is binding means that $v_{t+\Delta t} = \overline{v}$ and $v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v \geq \overline{v}$ (because otherwise $v_{t+\Delta t} < \overline{v}$, and hence, the constraint does not bind). Therefore, the exponential term $\exp(v_t - v_{t+\Delta t})$ in equation (A10) can be replaced with $\exp(\max\{0, v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\})$. When the constraint of investor $A$ does not bind the latter term vanishes and we obtain equation (A8). Therefore, both equations (A8) and (A10) can be conveniently summarized by equation (29) for $\xi_{A,t+\Delta t}/\xi_{At}$.

2) The equation (30) for the price-dividend ratio can be easily obtained by substituting $S_t = (1 - l_A - l_B)\Psi_t$ into equation (14) for stock prices in terms of SPD and then dividing both sides by $D_t$. The equation (31) for the wealth-aggregate consumption ratio can be obtained by substituting $W_{it} = D_t\Phi_{it}$ into the equation for the static budget constraint in (18) and dividing both sides by $D_t$.

To derive the matrix of asset returns, we rewrite the stock price dynamics as follows:

$$\frac{\Delta S_t + D_{t+\Delta t}\Delta t}{S_t} = \mu_t\Delta t + \sigma_t\Delta w_t + J_t\Delta j_t.$$

Therefore, the matrix of time-$(t + \Delta t)$ bond, stock and insurance returns is given by:

$$\begin{pmatrix} 1 + r_t\Delta t & 1 + \mu_t\Delta t + \sigma_t\sqrt{\Delta t} & 0 \\ 1 + r_t\Delta t & 1 + \mu_t\Delta t - \sigma_t\sqrt{\Delta t} & 0 \\ 1 + r_t\Delta t & J_t & 1/P_t \end{pmatrix}.$$

It is easy to see that the determinant of the above matrix is given by $2\sigma_t\Delta t(1 + r_t\Delta t)/P_t$. Therefore, the matrix is non-degenerate when $\sigma_t \neq 0$.

3) In the unconstrained economy, the state variable $v_t^{unc}$ follows dynamics:

$$v_t^{unc} = \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t. \tag{A11}$$

Define processes $U_{t+\Delta t} = U_t + \Delta U_t$ and $V_{t+\Delta t} = V_t + \Delta V_t$, where increments are given by:

$$\Delta U_t = \max\{0; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\}, \quad \Delta V_t = \max\{0; \underline{v} - v_t - \mu_v\Delta t - \sigma_v\Delta w_t - J_v\Delta j_t\}. \tag{A12}$$

The process for the state variable in the constrained economy can be rewritten as

$$v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t + \Delta V_t - \Delta U_t. \tag{A13}$$

If the state variables have the same value at time 0, i.e. $v_0 = v_0^{unc}$, we obtain:

$$v_t = v_t^{unc} + V_t - U_t \tag{A14}$$

Next, we prove that the SPD is higher in the constrained economy.

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left( \frac{s(v_{t+\Delta t})}{s(v_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \exp(\Delta U_t), \tag{A15}$$

$$\frac{\xi^{unc}_{A,t+\Delta t}}{\xi^{unc}_{At}} = e^{-\rho \Delta t} \left( \frac{s(v^{unc}_{t+\Delta t})}{s(v^{unc}_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A}. \tag{A16}$$

Iterating the above equations, we obtain:

$$\frac{\xi_{At}}{\xi_{A0}} = e^{-\rho t} \left( \frac{s(v_t)}{s(v_0)} \frac{D_t}{D_0} \right)^{-\gamma_A} \exp(U_t),$$

$$\frac{\xi^{unc}_{At}}{\xi^{unc}_{A0}} = e^{-\rho t} \left( \frac{s(v^{unc}_t)}{s(v_0)} \frac{D_t}{D_0} \right)^{-\gamma_A}.$$

By the definition of $s(v)$ in equation (21), we have $e^v = (1 - s(v))^{\gamma_B} \cdot s(v)^{-\gamma_A}$. Hence,

$$\begin{aligned} \frac{\xi_{At}/\xi_{A0}}{\xi^{unc}_{At}/\xi^{unc}_{A0}} &= \left( \frac{s(v_t)}{s(v^{unc}_t)} \right)^{-\gamma_A} \exp(U_t) \\ &= \left( \frac{s(v^{unc}_t + V_t - U_t)}{s(v^{unc}_t)} \right)^{-\gamma_A} e^{v^{unc}_t} e^{-(v^{unc}_t - U_t)} \\ &\geq s(v^{unc}_t - U_t)^{-\gamma_A} e^{-(v^{unc}_t - U_t)} \cdot s(v^{unc}_t)^{\gamma_A} e^{v^{unc}_t} \\ &= (1 - s(v^{unc}_t - U_t))^{-\gamma_B} \cdot (1 - s(v^{unc}))^{\gamma_B} \geq 1. \end{aligned} \tag{A17}$$

Therefore, we conclude that $\xi_{At}/\xi_{A0} > \xi^{unc}_{At}/\xi^{unc}_{A0}$. The latter inequality and the equation for stock prices (14) imply that $\Psi(v_0) \geq \Psi^{unc}(v_0)$. The proof for the case when time-$t$ variables in the constrained and unconstrained economies coincide is analogous. ■

**Proof of Lemma 2.** Define the following function in discrete time:

$$\widehat{\Psi}(v_t; \theta) = \mathbb{E}^A_t \left[ e^{-\rho \Delta t + \Delta U_t} \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \right] + s(v_t)^\theta \Delta t, \tag{A18}$$

where $\Delta U_t$ is given by equation

$$\Delta U_t = \max\{0; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \overline{v}\}. \tag{A19}$$

Comparing equation (A18) with equations (30) and (31) for $\Psi$ and $\Phi_i$ and using the linearity of equation (A18), it easy to observe that $\Psi(v_t)$ and $\Phi_i(v_t)$ are given by the following equations in terms of $\widehat{\Psi}(v_t;\theta)$:

$$\Psi(v_t) = \widehat{\Psi}(v_t, -\gamma_A)s(v_t)^{-\gamma_A} - \Delta t,$$
$$\Phi(v_t) = \Big((\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}})\widehat{\Psi}(v; 1-\gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i)\widehat{\Psi}(v; -\gamma_A)\Big)\, s(v)^{\gamma_A}.$$

Taking limit $\Delta t \to 0$, we obtain equations (32) and (33) for $\Psi(v_t)$ and $\Phi_i(v_t)$.

First, we derive the equation for $\widehat{\Psi}(v_t;\theta)$ when $v_t$ belongs to the interior $(\underline{v}, \overline{v})$. For a sufficiently small $\Delta t$ we have $\Delta U_t = 0$, where $\Delta U_t$ is given by (A19). Then, we rewrite the expectation of $(D_{t+\Delta t})/D_t)^{1-\gamma_A}\widehat{\Psi}(v_t;\theta)$ as follows:

$$\begin{aligned}\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] &= (1-\lambda\Delta t)\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\Big|\text{normal}\right] \\ &\quad + \lambda\Delta t\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\Big|\text{crisis}\right].\end{aligned} \tag{A20}$$

Noting that in the crisis $D_{t+\Delta t}/D_t = 1 + \mu_v\Delta t + J_D$ and $v_{t+\Delta t} = \max\{\underline{v}; v_t + \mu_v\Delta t + J_v\}$ and in the normal state $D_{t+\Delta t}/D_t = 1 + \mu_D\Delta t + \sigma_D\Delta w_t$ and $v_{t+\Delta t} = v_t + \mu_v\Delta t + \sigma_v\Delta w_t$, using Taylor expansions for $(D_{t+\Delta t}/D_t)^{1-\gamma_A}$ and $\widehat{\Psi}(v_{t+\Delta t};\theta)$, we find:

$$\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{crisis}\right] = (1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v}; v_t + J_v\};\theta\Big). \tag{A21}$$

$$\begin{aligned}\mathbb{E}^A_t\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{normal}\right] &= \left[1 + \left((1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A\sigma_D^2}{2}\right)\Delta t\right]\widehat{\Psi}(v_t;\theta) \\ + \Big(\mu_v + (1-\gamma_A)\sigma_D\sigma_v\Big)\widehat{\Psi}'(v_t;\theta)\Delta t &+ \frac{\sigma_v^2}{2}\widehat{\Psi}''(v_t;\theta)\Delta t + o(\Delta t).\end{aligned} \tag{A22}$$

Substituting (A21)-(A22) into (A18), we obtain:

$$\begin{aligned}\widehat{\Psi}(v_t;\theta) &= \left[1 - \left(\lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\Delta t\right]\widehat{\Psi}(v_t;\theta) \\ &\quad + \Big(\mu_v + (1-\gamma_A)\sigma_D\sigma_v\Big)\widehat{\Psi}'(v;\theta)\Delta t + \frac{\sigma_v^2}{2}\widehat{\Psi}''(v;\theta)\Delta t \\ &\quad + \lambda(1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v}; v_t + J_v\};\theta\Big)\Delta t + s(v)^\theta\Delta t + o(\Delta t).\end{aligned} \tag{A23}$$

Canceling similar terms, diving by $\Delta t$, taking limit $\Delta t \to 0$, and noting that $\mu_v$, $\sigma_v$ and $J_v$ converge to $\widehat{\mu}_v$, $\widehat{\sigma}_v$ and $\widehat{J}_v$ given by (36)-(38), we obtain equation (34) for $\widehat{\Psi}(v_t;\theta)$.

Next, we derive the boundary conditions for $\widehat{\Psi}(v_t;\theta)$. From equation (25), the state variable dynamics at lower bound is $v_{t+\Delta t} = \underline{v} + \max\{0, \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}$. We use

$\Delta v_t$ to denote the difference of $v_{t+\Delta t}$ and $v_t$. In this case,

$$\Delta v_t = \max\{0, \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}. \tag{A24}$$

For sufficiently small $\Delta t$ increment $\Delta v_t$ is positive only in state $\omega_1$ and is zero otherwise. In state $\omega_1$, $\Delta v_t = \mu_v \Delta t + \sigma_v \sqrt{\Delta t}$. Therefore, the order of $\mathbb{E}_t^A [\Delta v_t]$ is $\sqrt{\Delta t}$, but second order terms involving $\Delta v_t$ have lower order:

$$\lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A [\Delta v_t]}{\sqrt{\Delta t}} = \frac{\widehat{\sigma}_v}{2},$$

$$\lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A \left[(\Delta v_t)^2\right]}{\sqrt{\Delta t}} = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A [\Delta v_t \Delta t]}{\sqrt{\Delta t}} = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A [\Delta v_t \Delta w_t]}{\sqrt{\Delta t}} = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t^A [\Delta v_t \Delta j_t]}{\sqrt{\Delta t}} = 0. \tag{A25}$$

Taylor expansion of $\widehat{\Psi}(v_{t+\Delta t}; \theta)$ at $v_t = \underline{v}$ is given by

$$\widehat{\Psi}(v_{t+\Delta t}; \theta) = \widehat{\Psi}(\underline{v}; \theta) + \widehat{\Psi}'(\underline{v}; \theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v}; \theta)\Delta v_t^2 + o(\sqrt{\Delta t}). \tag{A26}$$

In subsequent calculations we keep terms with order of $\sqrt{\Delta t}$ and higher. Using the above results, we obtain the following expansion:

$$\begin{aligned} &\mathbb{E}_t^A \left[ \left( \frac{D_{t+\Delta t}}{D_t} \right)^{1-\gamma_A} \widehat{\Psi}(v_{t+\Delta t}; \theta) \right] \\ &= \mathbb{E}_t^A \left[ (1 + \mu_D \Delta t + \sigma_D \Delta w_t + J_v \Delta j_t)^{1-\gamma_A} \left( \widehat{\Psi}(\underline{v}; \theta) + \widehat{\Psi}'(\underline{v}; \theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v}; \theta)\Delta v_t^2 \right) \right] \\ &= \widehat{\Psi}(\underline{v}; \theta) + \widehat{\Psi}'(\underline{v}; \theta)\mathbb{E}_t^A[\Delta v_t] + o(\sqrt{\Delta t}). \end{aligned} \tag{A27}$$

Substituting (A27) into (A18), taking into account that $\Delta U_t = 0$ at $v_t = \underline{v}$, and canceling $\widehat{\Psi}(\underline{v}; \theta)$ on both sides, we obtain the first boundary condition $\widehat{\Psi}'(\underline{v}; \theta) = 0$.

At the upper bound $v_t = \overline{v}$ investor $A$ is constrained, and hence, $\Delta U_t$ in (A19) is positive. From (25) the state variable dynamics at the upper bound is

$$v_{t+\Delta t} = \min\{\overline{v}, v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t - \Delta U_t. \tag{A28}$$

The order of $\mathbb{E}_t^A [\Delta U_t]$ is $\sqrt{\Delta t}$, but second order terms involving $\Delta U_t$ have lower order. Proceeding in the same way as (A25)-(A27), we arrive at

$$\widehat{\Psi}(\overline{v}; \theta) = \widehat{\Psi}(\overline{v}; \theta) + \left[ \widehat{\Psi}(\overline{v}; \theta) - \widehat{\Psi}'(\overline{v}; \theta) \right] \mathbb{E}_t^A [\Delta U_t] + o(\sqrt{\Delta t}).$$

Canceling similar terms taking the limit $\Delta t \to 0$, we obtain the second boundary condition $\widehat{\Psi}(\overline{v}; \theta) - \widehat{\Psi}'(\overline{v}; \theta) = 0$. ■

**Proof of Proposition 3.** 1) First, we solve the differential-difference equation in Lemma 2. To facilitate the derivations, we denote $g(x) = \widehat{\Psi}(x + \underline{v}; \theta)$ and apply the following changes of variables:

$$\begin{aligned} &x = v - \underline{v}, \quad \tilde{\sigma} = \widehat{\sigma}_v, \quad \tilde{\mu} = \widehat{\mu}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v, \quad \tilde{J} = -\widehat{J}_v, \quad \tilde{\lambda} = \lambda(1+J_D)^{1-\gamma_A}, \\ &\tilde{\rho} = \lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2. \end{aligned} \tag{A29}$$

Equations (34) and (35) with new variables now become:

$$\frac{\tilde{\sigma}^2}{2} g''(x) + \tilde{\mu} g'(x) - \tilde{\rho} g(x) + \tilde{\lambda} g(\max\{x - \tilde{J}, 0\}) + s(x+\underline{v})^\theta = 0, \tag{A30}$$

$$g'(0) = 0, \quad g(\overline{v} - \underline{v}) - g'(\overline{v} - \underline{v}) = 0. \tag{A31}$$

Let $\mathcal{L}[g(x)] = \int_0^\infty e^{-zx} g(x) dx$ be the Laplace transform of $g(x)$, and similarly for other functions. The Laplace transforms of $g'(x)$, $g''(x)$ and $g(\max\{x - \tilde{J}, 0\})$ are related to $\mathcal{L}[g(x)]$ as follows:

$$\begin{aligned} \mathcal{L}[g'(x)] &= z\mathcal{L}[g(x)] - g(0), \\ \mathcal{L}[g''(x)] &= z^2\mathcal{L}[g(x)] - zg(0) - g'(0), \\ \mathcal{L}\left[g(\max\{x - \tilde{J}, 0\})\right] &= \int_0^\infty e^{-zx} g(\max\{x - \tilde{J}, 0\}) dx \\ &= \int_0^{\tilde{J}} e^{-zx} g(0) dx + \int_{\tilde{J}}^\infty e^{-zx} g(x - \tilde{J}) dx \\ &= \frac{1}{z}(1 - e^{-\tilde{J}z}) g(0) + e^{-\tilde{J}z}\mathcal{L}[g(x)]. \end{aligned} \tag{A32}$$

Applying the transform to equation (A30), we arrive at the following equation:

$$\begin{aligned} &\frac{\tilde{\sigma}^2}{2}\left(z^2\mathcal{L}[g(x)] - zg(0) - g'(0)\right) + \tilde{\mu}\left(z\mathcal{L}[g(x)] - g(0)\right) - \tilde{\rho}\mathcal{L}[g(x)] \\ &\qquad + \tilde{\lambda}\left(e^{-\tilde{J}z}\mathcal{L}[g(x)] + \frac{1}{z}(1 - e^{-\tilde{J}z}) g(0)\right) + \mathcal{L}\left[s(x+\underline{v})^\theta\right] = 0. \end{aligned} \tag{A33}$$

Applying boundary condition $g'(0) = 0$ and solving for $\mathcal{L}[g(x)]$, we obtain:

$$\mathcal{L}[g(x)] = \frac{\mathcal{L}\left[s(x+\underline{v})^\theta\right]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} + g(0)\left(\frac{1}{z} - \frac{\tilde{\rho} - \tilde{\lambda}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z}\right). \tag{A34}$$

Now define a new function $\widehat{\psi}(x)$ through inverse Laplace transform

$$\widehat{\psi}(x) = \mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}}\right]. \tag{A35}$$

Next, we apply inverse transform to each term in (A34). Noting that $\mathcal{L}^{-1}[1/z] = 1$ and using the theorem which states that Laplace transform of a convolution is the product of Laplace transforms, we derive the following inverse transforms:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[s(x+\underline{v})^{\theta}\right]}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}}\right] &= \int_0^x s(y+\underline{v})^{\theta}\cdot\widehat{\psi}(x-y)dy, \\
\mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}}\cdot\frac{1}{z}\right] &= \int_0^x \mathbf{1}_{\{y\geq 0\}}\cdot\widehat{\psi}(x-y)dy = \int_0^x \widehat{\psi}(y)dy.
\end{aligned} \tag{A36}$$

The linearity of the Laplace transform gives the following equation:

$$g(x) = \mathcal{L}^{-1}\left[\mathcal{L}\left[g(x)\right]\right] = \int_0^x s(y+\underline{v})^{\theta}\cdot\widehat{\psi}(x-y)dy + g(0)\left[1-\left(\tilde{\rho}-\tilde{\lambda}\right)\int_0^x\widehat{\psi}(y)dy\right]. \tag{A37}$$

We calculate $g(0)$ below, and then after changing the variable back from $x$ to $v = x+\underline{v}$, substituting in expressions for $\tilde{\rho}$ and $\tilde{\lambda}$ from (A29), we obtain (39).

Next, we solve for $\widehat{\psi}(x)$ in closed form. We expand $\mathcal{L}\left[\widehat{\psi}(x)\right]$ as series, and sum up the inverse transforms of each term in the summation to get $\widehat{\psi}(x)$.

$$\begin{aligned}
\mathcal{L}\left[\widehat{\psi}(x)\right] &= \frac{1}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2-\tilde{\lambda}e^{-\tilde{J}z}} \\
&= (\tilde{\rho}-\tilde{\mu}z-\tfrac{\tilde{\sigma}^2}{2}z^2)^{-1}\cdot(1-\frac{\tilde{\lambda}e^{-\tilde{J}z}}{\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2})^{-1} \\
&= \sum_{n=0}^{\infty}\frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}.
\end{aligned} \tag{A38}$$

The above series converges for $z$ such that $|\tilde{\rho}-\tilde{\mu}z-(\tilde{\sigma}^2/2)z^2| > |\tilde{\lambda}\exp(-\tilde{J}z)|$. This holds if the real part of $z$ is sufficiently large, e.g., $\Re(z) > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho}+\tilde{\lambda}}$. The inverse Laplace transform can then be calculated along the line $(\overline{z}-i\infty, \overline{z}+i\infty)$ in the complex domain where $\overline{z} > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho}+\tilde{\lambda}}$, and hence, the inequality for $\Re(z)$ is satisfied.

Let $\zeta_- < \zeta_+$ be roots of $\tilde{\rho}-\tilde{\mu}z-\tilde{\sigma}^2z^2/2 = 0$, given by (44). We use the following inversion formula for $1/[(z-\zeta_+)(z-\zeta_-)]^{n+1}$ from page 1117 of Gradshteyn and Ryzhik (2007):

$$\mathcal{L}^{-1}\left[\frac{1}{[(z-\zeta_+)(z-\zeta_-)]^{n+1}}\right] = \frac{\sqrt{\pi}}{\Gamma(n+1)}\frac{x^{n+\frac{1}{2}}}{(\zeta_+-\zeta_-)^{n+\frac{1}{2}}}e^{\frac{\zeta_++\zeta_-}{2}x}I_{n+\frac{1}{2}}\left(\frac{\zeta_+-\zeta_-}{2}x\right). \tag{A39}$$

Function $e^{-n\tilde{J}z}$ in the complex domain corresponds to a shift from $x$ to $x-n\tilde{J}$. Therefore,

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho}-\tilde{\mu}z-\frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}\right] &= \tilde{\lambda}^n\left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1}\mathbf{1}_{x\geq n\tilde{J}} \\ &\times\frac{\sqrt{\pi}}{\Gamma(n+1)}\frac{(x-n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+-\zeta_-)^{n+\frac{1}{2}}}e^{\frac{\zeta_++\zeta_-}{2}(x-n\tilde{J})}I_{n+\frac{1}{2}}\left(\frac{(\zeta_+-\zeta_-)(x-n\tilde{J})}{2}\right).\end{aligned} \tag{A40}$$

Consequently, the explicit expression for $\widehat{\psi}(x)$ is given by:

$$\widehat{\psi}(x)=\sum_{n=0}^{\infty}\tilde{\lambda}^n\left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1}\frac{\mathbf{1}_{\{x\geq n\tilde{J}\}}\sqrt{\pi}}{\Gamma(n+1)}\frac{(x-n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+-\zeta_-)^{n+\frac{1}{2}}}e^{\frac{\zeta_++\zeta_-}{2}(x-n\tilde{J})}I_{n+\frac{1}{2}}\left(\frac{(\zeta_+-\zeta_-)(x-n\tilde{J})}{2}\right). \tag{A41}$$

where function $I_{n+\frac{1}{2}}(\cdot)$ is a modified Bessel function of the first kind, $\zeta_-<\zeta_+$ are given by (44) and $\tilde{\rho}$, $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{\lambda}$, and $\tilde{J}$ are defined in (A29). Bessel function $I_{n+\frac{1}{2}}(\cdot)$ is given by (see equation 8.467 in Gradshteyn and Ryzhik (2007)):

$$I_{n+\frac{1}{2}}(z)=\frac{1}{\sqrt{2\pi z}}\left[e^z\sum_{m=0}^{n}\frac{(-1)^m(n+m)!}{m!(n-m)!(2z)^m}+(-1)^{n+1}e^{-z}\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!(2z)^m}\right]. \tag{A42}$$

Substituting (A42) into (A41), after minor algebra, we obtain expression (41) for $\widehat{\psi}(x)$. We note, that for a fixed $x$ the summation in (A41) is finite, and hence well-defined, due to the presence of indicator functions.

To find $g(0)$ in equation (A37), we first evaluate $\widehat{\psi}(0)$. From the above formula (A41), because $\mathbf{1}_{\{x\geq n\tilde{J}\}}$ is zero for any $n$ greater than 0, we obtain

$$\widehat{\psi}(0)=-\frac{2}{\tilde{\sigma}^2}\cdot\frac{e^{\zeta_+\cdot 0}-e^{\zeta_-\cdot 0}}{\zeta_+-\zeta_-}=0. \tag{A43}$$

Differentiating (A37) and using $\widehat{\psi}(0)=0$, we find:

$$g'(x)=\int_0^x s(y+\underline{v})^\theta\cdot\widehat{\psi}'(x-y)dy-g(0)\cdot\left(\tilde{\rho}-\tilde{\lambda}\right)\widehat{\psi}(x), \tag{A44}$$

We solve for $g(0)$ from the boundary condition $g(\overline{v}-\underline{v})-g'(\overline{v}-\underline{v})=0$ and obtain:

$$g(0)=\frac{\int_0^{\overline{v}-\underline{v}}s(y+\underline{v})^\theta\cdot\left[\widehat{\psi}'(\overline{v}-\underline{v}-y)-\widehat{\psi}(\overline{v}-\underline{v}-y)\right]dy}{1-\left(\tilde{\rho}-\tilde{\lambda}\right)\int_0^{\overline{v}-\underline{v}}\widehat{\psi}(y)dy+\left(\tilde{\rho}-\tilde{\lambda}\right)\widehat{\psi}(\overline{v}-\underline{v})}. \tag{A45}$$

Substituting (A45) into (A37), we derive equation (39) for $\widehat{\Psi}(v;\theta)$.

2) Next we solve for stock volatility and jump size. In the unconstrained region $\underline{v} < v_t < \overline{v}$, stock price $S_t$, dividend $D_t$ and state variable $v_t$ follow processes:

$$\begin{aligned} dS_t &= S_t[\mu_t d_t + \sigma_t dw_t + J_t dj_t], \\ dD_t &= D_t[\mu_D d_t + \sigma_D dw_t + J_D dj_t], \\ dv_t &= \widehat{\mu}_v dt + \widehat{\sigma}_v dw_t + \left(\max\{\underline{v}; v_t + \widehat{J}_v\} - v_t\right) dj_t. \end{aligned} \tag{A46}$$

Applying Ito's lemma to $S_t = \widehat{\Psi}(v_t; -\gamma_A) s(v_t)^{-\gamma_A}$, and matching $dw_t$ and $dj_t$ terms, after some algebra, we obtain $\sigma_t$ and $J_t$ in Proposition 3.

Equation equation (9) for $W_{i,t+\Delta t}$, implies the following expressions for $n^*_{i,st}$ and $b^*_{it}$:

$$\begin{aligned} n^*_{i,st} &= \sqrt{\frac{\mathrm{var}_t[W_{i,t+\Delta t} - W_{it}|\text{normal}]}{\mathrm{var}_t[\Delta S_t + (1 - l_A - l_B)D_t\Delta t|\text{normal}]}}, \\ b^*_{it} &= \mathbb{E}_t[W_{i,t+\Delta t|}\text{normal}] - n_{it}\mathbb{E}_t[S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t|\text{normal}]. \end{aligned}$$

Taking $\Delta t \to 0$ limit in the above expressions and using expansions similar to those in the proof of Lemma 2, we obtain the number of stocks and the leverage per the market value of stocks in equation (47).

3) Here, we prove the existence of $\underline{v}$ and $\overline{v}$. Boundaries $\underline{v}$ and $\overline{v}$ are solutions to equations (23), which we now seek to express in terms functions $\widehat{\Psi}(v; \theta)$. Substituting (32) and (33) into (23), dividing both sides by $s(v)^{\gamma_A}$ and taking the limit $\Delta t \to 0$, we obtain:

$$\begin{aligned} \widehat{\Psi}(\overline{v}; 1 - \gamma_A) - l_A\widehat{\Psi}(\overline{v}; -\gamma_A) &= k_A(1 - l_A - l_B)\widehat{\Psi}(\overline{v}; -\gamma_A) \\ (1 - l_B)\widehat{\Psi}(\underline{v}; -\gamma_A) - \widehat{\Psi}(\underline{v}; 1 - \gamma_A) &= k_B(1 - l_A - l_B)\widehat{\Psi}(\underline{v}; -\gamma_A). \end{aligned} \tag{A47}$$

Let $\tilde{l}_A = l_A + k_A(1 - l_A - l_B)$ and $\tilde{l}_B = l_B + k_B(1 - l_A - l_B)$. Because $k_A + k_B < 1$, it is easy to observe that $1 - \tilde{l}_B > \tilde{l}_A$, and hence, $\gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B) < \gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A)$.

Equations (A47) are equivalent to the following two equations:

$$\frac{\widehat{\Psi}(\overline{v}; 1 - \gamma_A)}{\widehat{\Psi}(\overline{v}; -\gamma_A)} = \tilde{l}_A, \quad \frac{\widehat{\Psi}(\underline{v}; 1 - \gamma_A)}{\widehat{\Psi}(\underline{v}; -\gamma_A)} = 1 - \tilde{l}_B. \tag{A48}$$

Define

$$L_B(\underline{v}, \overline{v}) = \frac{\widehat{\Psi}(\underline{v}; 1 - \gamma_A)}{\widehat{\Psi}(\underline{v}; -\gamma_A)}. \tag{A49}$$

Substituting $\widehat{\Psi}(v; \theta)$ from (39) into equation (A49), after some algebra, we obtain:

$$L_B(\underline{v}, \overline{v}) = \frac{\displaystyle\int_{\underline{v}}^{\overline{v}} \left[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\right] s(y)^{-\gamma_A} \cdot s(y) dy}{\displaystyle\int_{\underline{v}}^{\overline{v}} \left[\widehat{\psi}(\overline{v} - y) - \widehat{\psi}'(\overline{v} - y)\right] s(y)^{-\gamma_A} dy}. \tag{A50}$$

$L_B(\underline{v}, \overline{v})$ is a weighted average of a decreasing function $s(y)$ from $\underline{v}$ to $\overline{v}$. By (A63) in Lemma A.1 the weighting function $\left[\widehat{\psi}(\overline{v}-y) - \widehat{\psi}'(\overline{v}-y)\right] s(y)^{-\gamma_A}$ is positive. Consequently $L_B(\underline{v}, \overline{v}) < s(\underline{v})$ and the function is decreasing in its first argument because

$$\frac{\partial}{\partial \underline{v}} L_B(\underline{v}, \overline{v}) = \frac{\left[\widehat{\psi}(\overline{v}-\underline{v}) - \widehat{\psi}'(\overline{v}-\underline{v})\right] s(\underline{v})^{-\gamma_A}}{\int_{\underline{v}}^{\overline{v}} \left[\widehat{\psi}(\overline{v}-y) - \widehat{\psi}'(\overline{v}-y)\right] s(y)^{-\gamma_A} dy} \left[L_B(\underline{v}, \overline{v}) - s(\underline{v})\right] < 0. \tag{A51}$$

Consequently, for any $\overline{v} \geq \gamma_B \ln(1-\tilde{l}_A) - \gamma_A \ln(\tilde{l}_A)$,

$$L_B(\gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1-\tilde{l}_B), \overline{v}) < s(\gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1-\tilde{l}_B)) = 1 - \tilde{l}_B. \tag{A52}$$

Below, we prove that there exists a $\underline{V} < 0$ such that $L_B(\underline{V}, \overline{v}) > 1 - \tilde{l}_B$ for any $\overline{v} \geq \gamma_B \ln(1-\tilde{l}_A) - \gamma_A \ln(\tilde{l}_A)$. Consequently, by the intermediate value theorem, equation (A49) has a solution $\underline{v}$ for any fixed $\overline{v}$.

Using inequalities (A64) and (A65) from Lemma A.1 and inequality (A79) from Lemma A.2 below, we derive the following inequality:

$$\begin{aligned}
1 - L_B(\underline{V}, \overline{v}) &= \frac{\int_{\underline{V}}^{\overline{v}} \left[\widehat{\psi}(\overline{v}-y) - \widehat{\psi}'(\overline{v}-y)\right] s(y)^{-\gamma_A}(1-s(y))dy}{\int_{\underline{V}}^{\overline{v}} \left[\widehat{\psi}(\overline{v}-y) - \widehat{\psi}'(\overline{v}-y)\right] s(y)^{-\gamma_A} dy} \\
&< \frac{\int_{\underline{V}}^{\overline{v}} \left[-e^{z^+(\overline{v}-y)}\widehat{\psi}'(0)\right] (2^{\gamma_B+1}e^{y} + 2^{\gamma_A}e^{\frac{1}{\gamma_B}y})dy}{\int_{\underline{V}}^{\overline{v}-1} \left[-e^{z^+(\overline{v}-y-1)}(z^+-1)\widehat{\psi}(1)\right] s(\overline{v})^{-\gamma_A} dy} \\
&= \frac{\widehat{\psi}'(0)e^{z^+}s(\overline{v})^{\gamma_A}}{(z^+-1)\widehat{\psi}(1)} \cdot \frac{\int_{\underline{V}}^{\overline{v}} 2^{\gamma_B+1}e^{(1-z^+)y} + 2^{\gamma_A}e^{(\frac{1}{\gamma_B}-z^+)y}dy}{\int_{\underline{V}}^{\overline{v}-1} e^{-z^+y}dy} \\
&< \frac{\widehat{\psi}'(0)e^{z^+}s(\gamma_B \ln(1-\tilde{l}_A) - \gamma_A \ln(\tilde{l}_A))^{\gamma_A}}{(z^+-1)\widehat{\psi}(1)} \cdot \frac{\int_{\underline{V}}^{\infty} 2^{\gamma_B+1}e^{(1-z^+)y} + 2^{\gamma_A}e^{(\frac{1}{\gamma_B}-z^+)y}dy}{\int_{\underline{V}}^{\gamma_B \ln(1-\tilde{l}_A)-\gamma_A \ln(\tilde{l}_A)-1} e^{-z^+y}dy}.
\end{aligned} \tag{A53}$$

As $y$ decreases, the denominator term $e^{-z^+y}$ increases exponentially faster than any term on the numerator. Consequently, the right-hand side of the above inequality converges to 0 as $\underline{V} \to -\infty$, which can be formally verified by L'Hôpital's rule. Therefore, there exists a $\underline{V} < 0$ not dependent on $\overline{v}$ such that $1 - L_B(\underline{V}, \overline{v}) < \tilde{l}_B$, or, equivalently,

$$L_B(\underline{V}, \overline{v}) > 1 - \tilde{l}_B. \tag{A54}$$

For a given $\overline{v}$, $L_B(\underline{v}, \overline{v})$ is an continuously decreasing function of $\underline{v}$ that takes different signs at the endpoints of the interval $\left[\underline{V}, \gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B)\right]$. Therefore, by the intermediate value theorem there exists unique $\underline{v} \in \left[\underline{V}, \gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B)\right]$ such that $L_B(\underline{v}, \overline{v}) = 1 - \tilde{l}_B$, and this defines a mapping $\underline{v} = m_B(\overline{v})$. Since $L_B$ has non-zero partial derivative with respect to $\underline{v}$, $m_B(\cdot)$ is continuous by the implicit function theorem.

Similar to (A49), we define

$$L_A(\underline{v}, \overline{v}) = \frac{\widehat{\Psi}(\overline{v}; 1 - \gamma_A)}{\widehat{\Psi}(\overline{v}; -\gamma_A)}. \tag{A55}$$

Substituting $\widehat{\Phi}(v, \theta)$ from (39) into (A55), after some algebra, we obtain:

$$L_A(\underline{v}, \overline{v}) = \frac{\displaystyle\int_{\underline{v}}^{\overline{v}} \left[q'(\overline{v} - \underline{v})\widehat{\psi}(\overline{v} - y) - q(\overline{v} - \underline{v})\widehat{\psi}'(\overline{v} - y)\right] s(y)^{-\gamma_A} \cdot s(y) dy}{\displaystyle\int_{\underline{v}}^{\overline{v}} \left[q'(\overline{v} - \underline{v})\widehat{\psi}(\overline{v} - y) - q(\overline{v} - \underline{v})\widehat{\psi}'(\overline{v} - y)\right] s(y)^{-\gamma_A} dy}. \tag{A56}$$

Proceeding the same way as above, for any $\underline{v}$ less than or equal to $\gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B)$, there exists a $\overline{v} \in \left[\gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A), \overline{V}\right)$ that satisfies $L_A(\underline{v}, \overline{v}) = \tilde{l}_A$, where $\overline{V}$ does not depend on $\underline{v}$ .This defines a continuous mapping $\overline{v} = m_A(\underline{v})$.

Consider the following system of two equations with two unknowns:

$$\overline{v} = m_A(\underline{v}), \quad \underline{v} = m_B(\overline{v}), \tag{A57}$$

where $m_A(\cdot)$ maps $\underline{v} \in \left(-\infty, \gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B)\right]$ to $\overline{v} \in \left[\gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A), \overline{V}\right]$, and $m_B(\cdot)$ maps $\overline{v} \in \left[\gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A), \infty\right)$ to $\underline{v} \in \left[\underline{V}, \gamma_B \ln(\tilde{l}_B) - \gamma_A \ln(1 - \tilde{l}_B)\right]$. Consider now a composition function $m(v) \equiv m_A(m_B(v))$. Function $m(\cdot)$ maps $\overline{v} \in \left[\gamma_B \ln(1 - \tilde{l}_A) - \gamma_A \ln(\tilde{l}_A), \overline{V}\right]$ into itself. Because $m(v)$ is continuous, it has a fixed point $\overline{v}$ by the intermediate value theorem. Then, $\overline{v}$ and $\underline{v} \equiv m_B(\overline{v})$ satisfy equations (A57).

As demonstrated in Barro (2009), the price-dividend ratios in homogeneous-investor economies populated by investors $A$ and $B$, respectively, are given by:

$$\Psi_A = \frac{1}{\rho + (1 - \gamma_A)\mu_D + \dfrac{(1 - \gamma_A)\gamma_A}{2}\sigma_D^2 - \lambda(1 + J_D)^{1 - \gamma_A}}, \tag{A58}$$

$$\Psi_B = \frac{1}{\rho + (1 - \gamma_B)(\mu_D + \sigma_D \delta) + \dfrac{(1 - \gamma_B)\gamma_B}{2}\sigma_D^2 - \lambda_B(1 + J_D)^{1 - \gamma_B}}. \tag{A59}$$

After simple algebra, it can be shown that $\Psi_A = 1/(\tilde{\rho} - \tilde{\lambda})$ and $\Psi_A = 1/(\tilde{\rho} - \tilde{\mu} - 0.5\tilde{\sigma}^2 - \tilde{\lambda} e^{-\tilde{J}})$. Therefore, assumption (A61) in Lemma A.1 is equivalent to conditions $\Psi_A > 0$ and $\Psi_B > 0$

in Proposition 3. Moreover, investor's value functions are bounded because

$$\left|\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(\tau-t)}\frac{(c^*_{i\tau})^{1-\gamma_i}}{1-\gamma_i}d\tau\right]\right| = \left|\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(\tau-t)}\frac{s(v_\tau)^{1-\gamma_i}D_\tau^{1-\gamma_i}}{1-\gamma_i}d\tau\right]\right|$$

$$\leq \frac{\max\left\{s(\underline{v})^{1-\gamma_i}, s(\overline{v})^{1-\gamma_i}\right\}}{|1-\gamma_i|}\mathbb{E}_t\left[\int_t^{\infty} e^{-\rho(\tau-t)}D_\tau^{1-\gamma_i}d\tau\right] \tag{A60}$$

$$= \max\left\{s(\underline{v})^{1-\gamma_i}, s(\overline{v})^{1-\gamma_i}\right\}\frac{\Psi_i D_t^{1-\gamma_i}}{|1-\gamma_i|} < +\infty.$$

Finally, we note that our derivation of equilibrium assumes that payoff matrix is invertible, which is equivalent to $\sigma_t \neq 0$, as demonstrated in Proposition 2. ■

**Lemma A.1 (Inequalities for $\hat{\psi}(x)$ and $\hat{\psi}'(x)$).** *Suppose, the model parameters are such that the following two inequalities are satisfied:*

$$\tilde{\rho} - \tilde{\lambda} > 0, \quad \tilde{\rho} - \tilde{\mu} - \frac{\tilde{\sigma}^2}{2} - \tilde{\lambda}e^{-\tilde{J}} > 0, \tag{A61}$$

*where $\tilde{\rho}$, $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{\sigma}$ and $\tilde{J}$ are given by equations* (A29)*. Let function $q(x)$ be given by*

$$q(x) = 1 - (\tilde{\rho} - \tilde{\lambda})\int_0^x \widehat{\psi}(y)dy. \tag{A62}$$

*Then for all $x > 0$ and $\overline{v} > \underline{v}$ the following inequalities are satisfied:*

$$\begin{gathered}\widehat{\psi}(x) < 0, \quad \widehat{\psi}'(x) < 0,\\ \widehat{\psi}(x) - \widehat{\psi}'(x) > 0,\\ q'(\overline{v} - \underline{v})\widehat{\psi}(x) - q(\overline{v} - \underline{v})\widehat{\psi}'(x) > 0.\end{gathered} \tag{A63}$$

*Furthermore, there exists $z^+ > 1$ such that the following inequalities are satisfied:*

$$\widehat{\psi}(x) - \widehat{\psi}'(x) > -e^{z^+(x-1)}(z^+ - 1)\widehat{\psi}(1), \quad \text{for} \;\; x \geq 1, \tag{A64}$$

$$\widehat{\psi}(x) - \widehat{\psi}'(x) < -e^{z^+ x}\widehat{\psi}'(0), \quad \text{for} \;\; x > 0. \tag{A65}$$

**Proof of Lemma A.1.** From definition (A35), $\widehat{\psi}(x)$ satisfies equation:

$$\left[\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}\right]\mathcal{L}\left[\widehat{\psi}(x)\right] = 1. \tag{A66}$$

Dividing the above equation by $z$, applying inverse Laplace transform and using the fact that $\widehat{\psi}(0) = 0$, we find that $\widehat{\psi}(x)$ satisfies the following integro-differential equation:

$$\frac{\tilde{\sigma}^2}{2}\widehat{\psi}'(x) = -1 - \tilde{\mu}\widehat{\psi}(x) + (\tilde{\rho} - \tilde{\lambda})\int_0^x \widehat{\psi}(y)dy + \tilde{\lambda}\int_{\max\{x-\tilde{J},0\}}^x \widehat{\psi}(y)dy. \tag{A67}$$

Letting $x = 0$ in equation (A67), we obtain $\widehat{\psi}'(0) < 0$. Therefore, because $\widehat{\psi}(0) = 0$, $\widehat{\psi}(x) < 0$ in some neighborhood of 0. We first prove that $\widehat{\psi}(x) < 0$ for all $x > 0$. Suppose, on the contrary, that there exists $x > 0$ such that $\widehat{\psi}(x) \geq 0$. Let $\underline{x} = \inf\{x \in R^+ : \widehat{\psi}(x) \geq 0\}$. By the continuity of $\widehat{\psi}(x)$, we have $\widehat{\psi}(\underline{x}) = 0$ and $\widehat{\psi}(x) < 0$ for $x \in (0, \underline{x})$. Evaluating equation (A67) at $\underline{x}$, we obtain:

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\widehat{\psi}'(\underline{x}) &= -1 - \tilde{\mu}\widehat{\psi}(\underline{x}) + (\tilde{\rho} - \tilde{\lambda})\int_0^{\underline{x}} \widehat{\psi}(y)dy + \tilde{\lambda}\int_{\max\{\underline{x}-\tilde{J},0\}}^{\underline{x}} \widehat{\psi}(y)dy \\ &< -1 - \tilde{\mu}\cdot 0 + (\tilde{\rho} - \tilde{\lambda})\int_0^{\underline{x}} 0\cdot dy + \tilde{\lambda}\int_{\max\{\underline{x}-\tilde{J},0\}}^{\underline{x}} 0\cdot dy = -1.\end{aligned} \tag{A68}$$

The inequality (A68) is satisfied because $\tilde{\rho} - \tilde{\lambda} > 0$ by assumption (A61). However, $\widehat{\psi}'(\underline{x})$ being negative is inconsistent with $\underline{x}$ being the smallest positive number such that $\widehat{\psi}(\underline{x}) = 0$ because $\widehat{\psi}(x)$ cannot be a decreasing function at $\underline{x}$. Therefore, we arrive at a contradiction, and hence, $\widehat{\psi}(x) < 0$ for all $x > 0$.

Consider function $h(z) \equiv \tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}$. By assumption (A61), $h(0) > 0$ and $h(1) > 0$. It can be easily observed that $h(-\infty) = h(+\infty) = -\infty$. Therefore, by the intermediate value theorem there exist two real roots $z^- < 0$ and $z^+ > 1$ that satisfy equation $h(z) = 0$. Furthermore, function $h(z)$ is concave because $h''(z) < 0$. The concavity of $h(z)$ implies that $h(z) \geq 0$ for all $z \in [z^-, z^+]$.

Let $\widehat{z}$ be any number such that $\widehat{z} \in [z^-, z^+]$, and let $\widehat{\alpha}(x) \equiv e^{-\widehat{z}x}\widehat{\psi}(x)$. Next, we establish that $\widehat{\alpha}'(x) < 0$ for all $x \geq 0$. Differentiating equation (A67), we obtain:

$$\frac{\tilde{\sigma}^2}{2}\widehat{\psi}''(x) = -\tilde{\mu}\widehat{\psi}'(x) + \tilde{\rho}\widehat{\psi}(x) - \tilde{\lambda}\widehat{\psi}(x - \tilde{J})1_{x\geq\tilde{J}}. \tag{A69}$$

Substituting $\widehat{\psi}(x) = e^{\widehat{z}x}\widehat{\alpha}(x)$ into equation (A69), after some algebra, we find:

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\widehat{\alpha}''(x) &= -(\tilde{\mu} + \tilde{\sigma}^2\widehat{z})\widehat{\alpha}'(x) + (\tilde{\rho} - \tilde{\mu}\widehat{z} - \tfrac{\tilde{\sigma}^2}{2}\widehat{z}^2 - \tilde{\lambda}e^{-\tilde{J}\widehat{z}})\widehat{\alpha}(x) + \tilde{\lambda}e^{-\tilde{J}\widehat{z}}\left[\widehat{\alpha}(x) - \widehat{\alpha}(x - \tilde{J})1_{x\geq\tilde{J}}\right] \\ &= -(\tilde{\mu} + \tilde{\sigma}^2\widehat{z})\widehat{\alpha}'(x) + (\tilde{\rho} - \tilde{\mu}\widehat{z} - \tfrac{\tilde{\sigma}^2}{2}\widehat{z}^2 - \tilde{\lambda}e^{-\tilde{J}\widehat{z}})\int_0^x \widehat{\alpha}'(y)dy + \tilde{\lambda}e^{-\tilde{J}\widehat{z}}\int_{\max\{x-\tilde{J},0\}}^x \widehat{\alpha}'(y)dy.\end{aligned} \tag{A70}$$

We observe that $\widehat{\alpha}(0) = \widehat{\psi}(0) = 0$, $\widehat{\alpha}'(0) = -\widehat{z}\widehat{\psi}(0) + \widehat{\psi}'(0) < 0$ because $\widehat{\psi}(0) = 0$ and $\widehat{\psi}'(0) < 0$. The rest of the proof for $\widehat{\alpha}'(x) < 0$ is similar to that of $\widehat{\psi}(x) < 0$. Consequently, differentiating $\widehat{\alpha}(x)$ and dividing $\widehat{\alpha}'(x) < 0$ by $e^{-\widehat{z}x}$, we obtain:

$$\widehat{z}\widehat{\psi}(x) - \widehat{\psi}'(x) > 0, \text{ for any } \widehat{z} \in [z^-, z^+]. \tag{A71}$$

In particular for $\widehat{z} = 0$ we find $\widehat{\psi}'(x) < 0$, and for $\widehat{z} = 1$ we find $\widehat{\psi}(x) - \widehat{\psi}'(x) > 0$. Therefore, we have proven the first three inequalities in (A63).

Next, we prove (A64) and (A65). For $x > 1$, using inequality (A71) and the fact that $\widehat{\alpha}(x) = e^{-\widehat{z}x}\widehat{\psi}(x)$ is a decreasing function, we establish inequality (A64) as follows:

$$\begin{aligned}\widehat{\psi}(x) - \widehat{\psi}'(x) = (1 - z^+)\widehat{\psi}(x) + (z^+\widehat{\psi}(x) - \widehat{\psi}'(x)) &> -e^{z^+x}(z^+ - 1)(e^{-z^+x}\widehat{\psi}(x)) \\ &> -e^{z^+x}(z^+ - 1)(e^{-z^+1}\widehat{\psi}(1)).\end{aligned} \tag{A72}$$

To prove (A65), let $\tilde{\alpha}(x) = -e^{-z^+x}\widehat{\psi}'(x)$. Differentiating equation (A69) and rewriting it in terms of $\tilde{\alpha}(x)$, we derive the following equation:

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\tilde{\alpha}''(x) &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\tilde{\alpha}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\tilde{\alpha}(x) - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\tilde{\alpha}(\max\{x - \tilde{J}, 0\}) \\ &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\tilde{\alpha}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\int_{\max\{x-\tilde{J},0\}}^{x}\tilde{\alpha}'(y)dy + \left[\tilde{\lambda}e^{-\tilde{J}z^+} - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\right]\tilde{\alpha}(0).\end{aligned} \tag{A73}$$

Letting $x = 0$ in (A69), we find that $\widehat{\psi}''(0) = -2(\tilde{\mu}/\tilde{\sigma}^2)\widehat{\psi}'(0)$. Consequently,

$$\tilde{\alpha}(0) = -\widehat{\psi}'(0) > 0 \quad \tilde{\alpha}'(0) = -\widehat{\psi}''(0) + z^+\widehat{\psi}'(0) = \frac{2}{\tilde{\sigma}^2}(\tilde{\mu} + \frac{\tilde{\sigma}^2}{2}z^+)\widehat{\psi}'(0) < 0, \tag{A74}$$

where the last inequality hold because $z^+ > 1$ and $z^+(\tilde{\mu} + 0.5\tilde{\sigma}^2 z^+) = \tilde{\rho} - \tilde{\lambda}e^{-\tilde{J}z^+} > \tilde{\rho} - \tilde{\lambda} > 0$. Similar to the above, we show that $\tilde{\alpha}'(x) < 0$. Hence, we derive (A65) as follows:

$$\widehat{\psi}(x) - \widehat{\psi}'(x) < -\widehat{\psi}'(x) = e^{z^+x}\tilde{\alpha}(x) < e^{z^+x}\tilde{\alpha}(0) = -e^{z^+x}\widehat{\psi}'(0). \tag{A75}$$

Finally, we prove the last inequality in (A63). We define $\widehat{\beta}(x) = e^{-z^+x}q(x)$ and next prove that $\widehat{\beta}'(x) < 0$. Proceeding in the same way as above, we express equation (A67) first in terms of $q(x)$ and then in terms of $\widehat{\beta}(x)$:

$$\frac{\tilde{\sigma}^2}{2}q''(x) = -\tilde{\mu}q'(x) + \tilde{\rho}q(x) - \tilde{\lambda}q(\max\{x - \tilde{J}, 0\}), \tag{A76}$$

$$\begin{aligned}\tfrac{\tilde{\sigma}^2}{2}\widehat{\beta}''(x) &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\widehat{\beta}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\widehat{\beta}(x) - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\widehat{\beta}(\max\{x - \tilde{J}, 0\}) \\ &= -(\tilde{\mu} + \tilde{\sigma}^2 z^+)\widehat{\beta}'(x) + \tilde{\lambda}e^{-\tilde{J}z^+}\int_{\max\{x-\tilde{J},0\}}^{x}\widehat{\beta}'(y)dy + \left[\tilde{\lambda}e^{-\tilde{J}z^+} - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\right]\widehat{\beta}(0).\end{aligned} \tag{A77}$$

For $x = 0$ we observe that $\widehat{\beta}(0) = q(0) = 1$, $\widehat{\beta}'(0) = -z^+q(0) + q'(0) = -z^+q(0) - (\tilde{\rho} - \tilde{\lambda})\widehat{\psi}(0) = -z^+ < 0$. Moreover, it is easy to observe that $\left[\tilde{\lambda}e^{-\tilde{J}z^+} - \tilde{\lambda}e^{-\min\{x,\tilde{J}\}z^+}\right]\widehat{\beta}(0) \le 0$ for all $x$. Proceeding in the same way as above, we find that $\widehat{\beta}'(x) < 0$, and hence, $q'(x) < z^+q(x)$. Using the latter inequality and the fact that $\widehat{\psi}(x) < 0$, we derive the last inequality in (A63) as follows:

$$\begin{aligned}q'(\overline{v} - \underline{v})\widehat{\psi}(x) - q(\overline{v} - \underline{v})\widehat{\psi}'(x) &\ge z^+q(\overline{v} - \underline{v})\widehat{\psi}(x) - q(\overline{v} - \underline{v})\widehat{\psi}'(x) \\ &= q(\overline{v} - \underline{v})\left[z^+\widehat{\psi}(x) - \widehat{\psi}'(x)\right] > 0. \quad \blacksquare\end{aligned} \tag{A78}$$

**Lemma A.2 (Inequality for consumption shares).** *Let $s(v_t)$ denote the consumption share of investor A. Then, for all $v \in \mathbb{R}$ the following inequality is satisfied:*

$$s(v)^{-\gamma_A}(1-s(v)) \le 2^{\gamma_B+1}e^v + 2^{\gamma_A}e^{v/\gamma_B}. \tag{A79}$$

**Proof of Lemma A.2.** First, we rewrite equation (21) in the following equivalent form:

$$s(v)^{-\gamma_A}(1-s(v))^{\gamma_B} = e^v. \tag{A80}$$

When $\gamma_B \le 1$, from the above equation we obtain the following inequality:

$$s(v)^{-\gamma_A}(1-s(v)) \le s(v)^{-\gamma_A}(1-s(v))^{\gamma_B} = e^v. \tag{A81}$$

For $\gamma_B > 1$ and $1 - s(v) \ge 1/2$, we find that:

$$s(v)^{-\gamma_A}(1-s(v)) \le 2^{\gamma_B-1}s(v)^{-\gamma_A}(1-s(v))^{\gamma_B} = 2^{\gamma_B-1}e^v. \tag{A82}$$

Finally, for $\gamma_B > 1$ and $s(y) \ge 1/2$ we have the following inequality:

$$s(y)^{-\gamma_A}(1-s(y)) \le 2^{\gamma_A-\gamma_A/\gamma_B}s(y)^{-\gamma_A/\gamma_B}(1-s(y)) < 2^{\gamma_A}e^{v/\gamma_B}. \tag{A83}$$

Combining all the inequalities (A81)-(A83), we obtain inequality (A79). ■

**Proof of Proposition 4.** From equation (13) for the bond price and the fact that $1 = B_t(1 + r_t\Delta t)$ we find that the riskless interest rate $r_t$ is given by:

$$\begin{aligned} r_t &= \frac{1 - \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t} \\ &= \frac{1 - (1-\lambda\Delta t)\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}] - \lambda\Delta t\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t}, \end{aligned} \tag{A84}$$

where $\xi_{A,t+\Delta t}/\xi_{At}$ is given by equation (29). We separately calculate $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ and $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]$, and then take the limit $\Delta t \to 0$.

We start with the derivation of $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ when $\underline{v} < v_t < \overline{v}$, and hence, by continuity, for a sufficiently small $\Delta t$ the economy is unconstrained next period, so that $\underline{v} < v_{t+\Delta t} < \overline{v}$. In the unconstrained region $\Delta v_t = \widehat{\mu}_v\Delta t + \widehat{\sigma}_v\Delta w_t$ and the SPD is given by (A8). From the expression for the SPD, using expansions (A93) and (A95), we obtain:

$$\begin{aligned} \mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{normal}\right] &= \mathbb{E}_t\Big[\Big((1 + a_t\Delta v_t + b_t(\Delta v_t)^2\Big)\Big(1 - r_A\Delta t - \kappa_A\Delta w_t\Big)|\text{normal}\Big] + o(\Delta t) \\ &= \mathbb{E}_t\Big[1 + a_t\Delta v_t + b_t(\Delta v_t)^2 - r_A\Delta t - \kappa_A\Delta w_t - \kappa_A a_t\Delta v_t\Delta w_t\Big|\text{normal}\Big] + o(\Delta t) \\ &= 1 + a_t\widehat{\mu}_v\Delta t + b_t\widehat{\sigma}_v^2\Delta t - r_A\Delta t - \kappa_A a_t\widehat{\sigma}_v\Delta t + o(\Delta t). \end{aligned} \tag{A85}$$

Conditioning on the crisis state, we have:

$$\begin{aligned}\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{crisis}\right] &= (1-\rho\Delta t)(1+\mu_D\Delta t+J_D)^{-\gamma_A}\left(\frac{s(\max\{\underline{v}, v_t+\mu_v\Delta t+J_v\})}{s(v_t)}\right)^{-\gamma_A}\\ &= (1+J_D)^{-\gamma_A}\left(\frac{s(\max\{\underline{v}, v_t+\widehat{J}_v\})}{s(v_t)}\right)^{-\gamma_A}+o(\Delta t).\end{aligned} \tag{A86}$$

Substituting $a_t$ and $b_t$ from (A94) into equation (A85), and then substituting (A85) and (A86) into equation (A84), after simple algebra, we obtain the interest rate (48) for the case $\underline{v} < v_t < \overline{v}$.

Now, we derive $r_t$ at the boundaries $\underline{v}$ and $\overline{v}$. The SPD is given by (29). Using expansions (A93) and (A95), we obtain the following expansion:

$$\begin{aligned}\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{normal}\right] &= \mathbb{E}_t\Big[\Big((1+a_t\Delta v_t+b_t(\Delta v_t)^2\Big)\Big(1-r_A\Delta t-\kappa_A\Delta w_t\Big)\\ &\qquad\times(1+\Delta U_t+0.5(\Delta U_t)^2)|\text{normal}\Big]+o(\Delta t)\\ &= \mathbb{E}_t\Big[1+a_t\Delta v_t+b_t(\Delta v_t)^2-r_A\Delta t-\kappa_A\Delta w_t-\kappa_A a_t\Delta v_t\Delta w_t\\ &\quad+\Delta U_t-\kappa_A\Delta w_t\Delta U_t+a_t\Delta U_t\Delta v_t+0.5(\Delta U_t)^2\Big|\text{normal}\Big]+O(\Delta t),\end{aligned} \tag{A87}$$

where $\Delta U_t$ is given by equation (A19). Using equation (25) for the process $v_t$ and equation (A19) for $\Delta U_t$, for a fixed $v_t$ and sufficiently small $\Delta t$, we find that $\Delta v_t$ and $\Delta U_t$ at the boundaries are given by:

$$\Delta v_t = \begin{cases}\min(0, \mu_v\Delta t+\sigma_v\Delta w_t), & \text{if } v_t=\overline{v},\\ \max(0, \mu_v\Delta t+\sigma_v\Delta w_t), & \text{if } v_t=\underline{v},\end{cases} \tag{A88}$$

$$\Delta U_t = \begin{cases}0, & \text{if } v_t<\overline{v},\\ \max(0, \mu_v\Delta t+\sigma_v\Delta w_t), & \text{if } v_t=\overline{v},\end{cases} \tag{A89}$$

We note that for a sufficiently small $\Delta t$ the sign of $\mu_v\Delta t+\sigma_v\Delta w_t$ is solely determined by the second term $\sigma_v\Delta w_t$ because it has higher order of magnitude $\sqrt{\Delta t}$. Volatility $\sigma_v$ is positive because under our assumptions investor $A$ is more risk averse and more pessimistic. Using the latter observation, substituting equations (A88) and (A89) into equation (A87) and computing the expectation, we obtain:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{normal}\right] = 1+\begin{cases} \left(\dfrac{a_t(\mu_v-\kappa_A\sigma_v)}{2}+\dfrac{b_t\sigma_v^2}{2}+\dfrac{\mu_v+\kappa_A\sigma_v+\sigma_v^2}{2}-r_A\right)\Delta t \\ \qquad +\dfrac{\sigma_v(1-a_t)}{2}\sqrt{\Delta t}+O(\Delta t), \quad \text{if} \quad v_t=\underline{v}, \\ \left(\dfrac{a_t\mu_v-a_t\kappa_A\sigma_v+b_t\sigma_v^2}{2}\right)\Delta t+\dfrac{a_t\sigma_v}{2}\sqrt{\Delta t}+O(\Delta t), \quad \text{if} \quad v_t=\overline{v}. \end{cases} \tag{A90}$$

We note, that $\mathbb{E}_t\left[\xi_{A,t+\Delta t}/\xi_{At}\Big|\text{crisis}\right]$ is given by (A86). Substituting (A90) and (A86) into equation (A84) for the interest rate $r_t$, we obtain the expression for $r_t$ in (48) for the case when $v_t$ is at the boundary.

To obtain the risk premium, we first decompose stock returns as follows:

$$\frac{\Delta S_t+D_{t+\Delta t}\Delta t}{S_t}=\mu_t\Delta t+\sigma_t\Delta w_t+J_t\Delta j_t. \tag{A91}$$

Multiplying both sides of the above equation by $\xi_{A,t+\Delta t}/\xi_{At}$ and taking expectations on both sides, we obtain:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t+D_{t+\Delta t}\Delta t}{S_t}\right]=\mu_t\Delta t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right]+\sigma_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right]+J_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta j_t\right].$$

On the other hand, from the equation for stock price (14) we find that:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t+D_{t+\Delta t}\Delta t}{S_t}\right]=1-\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right].$$

Combining the last two equations and the equation (A84) for the interest rate, we obtain:

$$\mu_t-r_t=-\left(\sigma_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right]+J_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta j_t\right]\right)\frac{1+r_t\Delta t}{\Delta t}. \tag{A92}$$

Then, proceeding in the same way as with the calculation of interest rates and using similar expansions, we obtain equation (49) for the risk premium. ■

**Lemma A.3 (Useful expansions).**

1) *For small increment $\Delta v_t = v_{t+\Delta t} - v_t$ the ratio $\left(s(v_{t+\Delta t})/s(v_t)\right)^{-\gamma_A}$ has expansion:*

$$\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}=1+a_t\Delta v_t+b_t(\Delta v_t)^2+o(\Delta t), \tag{A93}$$

*where coefficients $a_t$ and $b_t$ are given by:*

$$a_t = \frac{(1-s_t)\Gamma_t}{\gamma_B}, \quad b_t = \frac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2 + \frac{1}{2\gamma_A^2\gamma_B^2}s_t(1-s_t)\Gamma_t^3, \tag{A94}$$

$\Gamma_t = \gamma_A\gamma_B/(\gamma_A(1-s)+\gamma_B s)$ *is the risk aversion of the representative investor and* $s_t$ *is consumption share of investor A that solves equation* (21).

2) *For the case $J_D = 0$, the SPD in a one-investor economy can be expanded as follows:*

$$e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A} = 1 - r_A\Delta t - \kappa_A\Delta w_t + o(\Delta t), \tag{A95}$$

*where $r_A$ and $\kappa_A$ are the riskless rate and the Sharpe ratio in an economy populated only by investor A, given by:*

$$r_A = \rho + \gamma_A\mu_D - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2, \quad \kappa_A = \gamma_A\sigma_D. \tag{A96}$$

**Proof of Lemma A.3.** 1) We expand the ratio on the left-hand side of (A93) using Taylor's formula, and observe that $a_t = (s(v_t)^{-\gamma_A})'/s(v_t)^{-\gamma_A}$ and $b_t = 0.5(s(v_t)^{-\gamma_A})''/s(v_t)^{-\gamma_A}$. Differentiating, we obtain the following expressions for $a_t$ and $b_t$:

$$a_t = -\gamma_A\frac{s'(v_t)}{s(v_t)}, \quad b_t = \frac{\gamma_A(1+\gamma_A)}{2}\left(\frac{s'(v_t)}{s(v_t)}\right)^2 - \frac{\gamma_A}{2}\frac{s''(v)}{s(v)}. \tag{A97}$$

To find derivatives $s'(v)$ and $s''(v)$, we differentiate equation (21) twice with respect to $v$, and obtain two equations for the derivatives:

$$1 = -\left(\frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t}\right)s'(v_t), \tag{A98}$$

$$0 = \left(\frac{\gamma_A}{s_t^2} - \frac{\gamma_B}{(1-s_t)^2}\right)(s'(v_t))^2 - \left(\frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t}\right)s''(v_t). \tag{A99}$$

Finding $s'(v)$ and $s''(v)$ from the system (A98)–(A99) and substituting them into expressions (A97) for coefficients $a_t$ and $b_t$, after some algebra, we obtain expressions (A94).

2) Substituting $D_{t+\Delta t}/D_t$ from (1) into equation (A95), after some algebra, we obtain:

$$\begin{aligned} e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right) &= e^{-\rho\Delta t}\left(1+\mu_D\Delta t + \sigma_D\Delta w_t\right)^{-\gamma_A} \\ &= (1-\rho\Delta t)\left(1 - \left(\gamma_A\mu_D - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2\right)\Delta t - \gamma_A\sigma_D\right) + o(\Delta t) \\ &= 1 - r_A\Delta t - \kappa_A\Delta w_t + o(\Delta t). \quad \blacksquare \end{aligned} \tag{A100}$$

**Proof of Proposition 5.** Consider a reflected arithmetic Brownian motion with reflecting boundaries $\underline{v}$ and $\overline{v}$, which follows dynamics $dv_t = \widehat{\mu}_v dt + \widehat{\sigma}_v dw_t$ when $\underline{v} < v_t < \overline{v}$, where $w_t$ is a Brownian motion. The transition density for this process has been derived in closed form in Veestraeten (2004), and is given by:

$$
\begin{aligned}
f_v(v,\tau;v_t,t) = {} & \frac{1}{\sqrt{2\pi\widehat{\sigma}_v^2(\tau-t)}} \sum_{n=-\infty}^{+\infty} \left[ \exp\left( -\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2} n(\overline{v}-\underline{v}) - \frac{\left(v - v_t - \widehat{\mu}_v(\tau-t) + 2n(\overline{v}-\underline{v})\right)^2}{2\widehat{\sigma}_v^2(\tau-t)} \right) \right. \\
& \left. + \exp\left( -\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(v_t - \underline{v} + n(\overline{v}-\underline{v})\right) - \frac{\left(v - v_t - \widehat{\mu}_v(\tau-t) + 2(v_t - \underline{v} + n[\overline{v}-\underline{v}])\right)^2}{2\widehat{\sigma}_v^2(\tau-t)} \right) \right] \\
& + \frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2} \sum_{n=0}^{+\infty} \left[ \exp\left( -\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(\overline{v} - v + n[\overline{v}-\underline{v}]\right) \right) \mathcal{N}\left( \frac{v_t + \widehat{\mu}_v(\tau-t) - v - 2(\overline{v} - v + n[\overline{v}-\underline{v}])}{\widehat{\sigma}_v\sqrt{\tau-t}} \right) \right. \\
& \left. - \exp\left( \frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(v - \underline{v} + n[\overline{v}-\underline{v}]\right) \right) \left( 1 - \mathcal{N}\left( \frac{v_t + \widehat{\mu}_v(\tau-t) - v + 2(v - \underline{v} + n[\overline{v}-\underline{v}])}{\widehat{\sigma}_v\sqrt{\tau-t}} \right) \right) \right],
\end{aligned}
\tag{A101}
$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution of a standard normal distribution. By $F_v(v,\tau;v_t,t) = \text{Prob}\{v_\tau \le v|v_t\}$ we denote the corresponding cumulative distribution function of $v$ conditional on observing $v_t$ at time $t$. We observe that $s_t = s(v_t)$ is a decreasing function of $v_t$ implicitly defined by equation (21), and hence there exists a well-defined inverse function, which can be obtained from the same equation (21), and is given by: $s^{-1}(x) = \gamma_B(1-s) - \gamma_A \ln(s)$. The cumulative distribution function of consumption share $s_\tau$ at time $\tau$ conditional on observing $s_t$ at time $t$ can then be found as follows:

$$
\begin{aligned}
F(x,\tau;s_t,t) &= \text{Prob}\{s_\tau \le x|s_t\} \equiv \text{Prob}\{s(v_\tau) \le x|s_t\} \\
&= 1 - \text{Prob}\{v_\tau \le s^{-1}(x)|v_t\} \\
&= 1 - \text{Prob}\{v_\tau \le \gamma_B \ln(1-x) - \gamma_A \ln(x)|v_t\} \\
&= 1 - F_v(\gamma_B \ln(1-x) - \gamma_A \ln(x), \tau; v_t, t).
\end{aligned}
\tag{A102}
$$

Substituting $v_t = \gamma_B \ln(1-s_t) - \gamma_A \ln(s_t)$ into (A102), differentiating CDF $F(x,\tau;s_t,t)$ with respect to $x$ and setting $x = s$, we find that the transition PDF for $s$ is given by:

$$
f(s,\tau;s_t,t) = \left(\frac{\gamma_A}{s} + \frac{\gamma_B}{1-s}\right) f_v\Big(\gamma_B \ln(1-s) - \gamma_A \ln(s), \tau; \gamma_B \ln(1-s_t) - \gamma_A \ln(s_t), t\Big), \tag{A103}
$$

where transition density $f_v(v,\tau;v_t,t)$ is given by equation (A101).

The stationary distribution of variable $v$, calculated in Veestraeten (2004), is given by:

$$f_v(v) = \frac{2\widehat{\mu}_v}{\widehat{\sigma}_v} \frac{\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)v\Big)}{\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)\overline{v}\Big) - \exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)\underline{v}\Big)}. \tag{A104}$$

Proceeding in the same way as for the derivation of transition PDF (A103), we obtain stationary PDF (50) for consumption share $s$. ■

**Proof of Corollary 1.** From the fact that $\gamma_A = \gamma_B = \gamma$ we obtain that $\widehat{\mu}_v = -0.5\delta^2$ and $\widehat{\sigma}_v = \delta$, where $\widehat{\mu}_v$ and $\widehat{\sigma}_v$ are defined in Proposition 5. Therefore, $2\widehat{\mu}/\widehat{\sigma}_v^2 = -1$. Substituting $2\widehat{\mu}/\widehat{\sigma}_v = -1$ into equation (50) for the PDF of consumption share $s$ we obtain PDF (51) for the case of equal risk aversions. ■

**Proof of Corollary 2.** The proof easily follows by substituting boundary conditions (35) into the equation (45) for volatility $\sigma_t$ at the boundary values $\underline{v}$ and $\overline{v}$. ■