Pricing Inflation linked bonds and hedging bond portfolios: a comparative analysis applied to French OAT indexed bonds

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Abstract

Using market prices of inflation-linked bonds and nominal bonds issued by the French Treasury, both the real and nominal zero coupon curves are estimated from January 1, 2013 to December 31, 2015. Several methods are applied to extract zero coupon bond prices: bootstrapping, a piecewise constant forward rates method, a cubic spline model, and the Nelson and Siegel smoothing model. Next, based on the estimated real and nominal curves, several methodologies to hedge bond portfolios comprising indexed and nominal bonds are tested. The hedging ratios used in these strategies are based on an arbitrage-free model of the term structure and on a traditional duration measure. Hedging methods are compared according to three perspectives: impact of the extraction method, hedging methodology, and holding period. It seems that the extraction procedure and hedging methodology do not impact the hedging error measurement, while the holding period plays a key role. This paper also generates evidence on the benefits of hedging portfolios against real interest rate and inflation risks rather than against nominal interest rate risks.

JEL Classification: E43, G01, G12

Keywords: Bond pricing, break-even inflation, interest rate risk hedging, nominal interest rates, real interest rates, Treasury Inflation-Protected Securities.

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1. Introduction

Inflation-linked bonds are a unique asset class in that they offer a nearly perfect hedge against inflation. They have been issued since the 1990s by several governments and have gained in popularity. Global volume in this bond class has increased tenfold over the past decade, with the United States, the United Kingdom, and France among the largest issuers of these securities. The French and the German governments pledged to issue 10% of their total debt market in this type of security. Over the past decade, these securities have shown favorable performance and lower volatility relative to other risk assets. However, in the Eurozone, they are now less attractively valued given the low level of inflation. Nevertheless, Treasury Inflation-Protected Securities (TIPS) are a useful investment vehicle for investors whose liabilities are closely tied to changes in inflation or wages.

The difference between nominal and real interest rates is commonly designated in literature as *break-even inflation* and is the focus of most of the literature on inflation-linked bonds. This concept has increasingly gained attention as a measure of investors' inflation expectations. However, break-even inflation is a noisy measure of expected inflation because it includes an inflation risk premium component, and potentially a liquidity premium. Pericoli (2014) and Chen, Liu and Cheng (2010) calculate a more accurate measure to estimate the break-even inflation rate: the zero coupon real rate is subtracted from the corresponding zero coupon nominal rate. Our study differs from the above-mentioned studies in that its goal is not to supply a measure of market expectations for monetary policy purposes, but to price inflation-linked bonds and to hedge bond portfolios.

Our study builds on prior and concurrent research from three different strands of the literature on inflation-linked bonds. The first focuses on the pricing of these assets. There are relatively few studies on this particular topic, since they must be conducted in countries with a history of such bonds, such as France. D'Amico, Kim and Wei (2014) and Chen *et al.* (2010) focus on this topic. Further, D'Amico, Kim and Wei (2014) underline the issue of the liquidity of this segment of the bond market and provide evidence that the potential illiquidity of this market can distort the information content of prices. The second strand is fairly recent and relies on studies using inflation-linked bonds, but also other inflation-linked assets, like inflation swaps. The third strand includes studies that explore the behavior of real interest rates, inflation expectations, and inflation risk premium without incorporating information from indexed bonds, for example, Ang, Bekaert and Wei (2008).

The present paper focuses first on the pricing of French inflation-linked bonds. For this purpose, we extract both the real and nominal zero-coupon bond yields using three methods: the bootstrapping procedure, a piecewise constant forward rates method, and the Nelson and Siegel smoothing model. In a second step, the time series of the estimated real and nominal bond yields are used to test two bond portfolio hedging methods: the first is

derived from the Jarrow and Yildirim model (2003), and the second is the Fisher and Well duration. Bond portfolios comprising both compounding nominal and inflation-linked bonds are built to test each method.
The estimate of the real term structure relies on the market prices of inflation-linked bonds issued by the French Treasury on a daily basis from January 1, 2013 to December 31, 2015. Two types of inflation-indexed bonds are issued: OAT€i and OATi. Both are government bonds, but OAT€i are indexed on the euro area Harmonized Index of Consumer Prices excluding tobacco (HICP), while OATi are indexed to the domestic French Consumer Price Index (CPI). Since the results of Pericoli (2014) do not differ when one compares the estimates separately obtained from the two different kinds of OAT, we use both of them to ensure a sufficient number of observations over the estimation period. To estimate the nominal term structure, the classic OAT and BTAN issued by the French Treasury are used.
This study generates evidence that the bootstrapping method tracks accurately both real and nominal bond prices. The other implemented methods are less efficacious; the Jarrow and Yildirim piecewise constant forward rates method ranks second, followed by the smoothing methods. As noted by Pericoli (2014) the smoothing methods are very effective in capturing the general shape of the real term structure or to extract interest rate expectations, while they are less effective for pricing. More specifically, the Nelson and Siegel (1987) model performs very poorly when the number of bonds is low.
This confirms the conclusion of Sercu and Wu (1997) that bond pricing models based on Heath, Jarrow and Morton (1992) and Vasicek (1977) identify bond mispricing more accurately than smoothing methods.
As regards hedging, the methodology employed has a relatively low impact. Using the hedge ratios of the Jarrow and Yildirim (2003) model or the Fisher and Weil(1971) duration does not yield different results on all the studied portfolios. The most important result is that inflation-linked bonds maintain their power of protection and improve bond portfolio protection, even in a context of low inflation.
This paper proceeds as follows. Section 2 presents the pricing models used and the theoretical foundations for the implemented hedging strategies. Section 3 presents the data and the results of the pricing estimation, as well as implementation of hedging strategies. Finally, Section 4 describes our conclusions.

2. Pricing and hedging methodology

This paper uses four alternative methodologies to estimate the term structure of zero coupon nominal interest rates and of zero coupon real interest rates. The methods used include two non-smoothing methods, bootstrapping and a piecewise constant forward rate curve proposed by Jarrow and Yildirim (2003), and two smoothing methods, a cubic spline model and the Nelson and Siegel (1987) term structure smoothing model. The cubic spline method was initially developed by Mcculloch (1975) and applied to conventional bonds (i.e., not indexed bonds). It was extended by Mcculloch and Kochin (2000) to inflation-linked bonds. The two non-smoothing methods used in this paper provide estimated bond prices without the modeling constraints imposed by the cubic spline and smoothing models.

2.1 *Piecewise Jarrow and Yildirim method and bootstrapping methods*

The notations used in this paper have the following meanings:
r_r: real interest rate;
r_n: nominal interest rate;
$P_n(t,\tau)$: price at time t of a nominal zero-coupon bond (i.e., nominal discount function of a coupon bond payoff) maturing at time $t+\tau$;
$P_r(t,\tau)$: price at time t of a real zero-coupon bond (i.e., real discount function) maturing at time $t+\tau$;
IF_t: indexation factor of an indexed bond, which is the ratio between the value of the harmonized CPI excluding tobacco at time t (I_t) and the value of the same index at bond issuance date t_0 (I_{t0}).
$B_n(t,m)$: price on date t of a conventional bond that periodically pays C euros, and C+VF euros on maturity date $t+m$; bond current price is equal to the sum of the present value of its payoffs, given by:

$$B_n(t,m)=\sum_{\tau=1}^{m} CP_n(t,\tau)+VFP_n(t,m) \qquad (1)$$

$B_{tips}(t,m)$: price at date t of an inflation-indexed bond, also designated commonly by Treasury Inflation Protected Security (TIPS) that periodically pays C units of the CPI on each date $t+\tau$ between t and $t+m$, and C+VF units on maturity date $t+m$. Taking into account the current value of the bond indexation factor IF_t, the bond price is equal to the sum of the present value of its nominal payoffs, given by:

$$B_{tips}(t,m)=IF_t\left[\sum_{\tau=1}^{m} CP_r(t,\tau)+VFP_r(t,m)\right] \qquad (2)$$

The various methodologies to estimate a term structure may rely on the price of a nominal zero coupon bond as a function of the forward rates until maturity as follows:

$$P_n(t,\tau)=\exp\left(-\int_t^{t+\tau} f_n(t,u)\,du\right) \qquad (3)$$

where $f_n(t,u)$ is the nominal forward rate for time $t+u$. The nominal spot rate or, equivalently, the yield-to-maturity of this zero coupon bond is:

$$r_n(t,\tau)=\frac{1}{\tau}\int_t^{t+\tau} f_n(t,u)\,du \qquad (4)$$

The equivalent representations for a real zero coupon bond and the corresponding real forward rates and spot rate are:

$$P_r(t,\tau)=\exp\left(-\int_t^{t+\tau} f_r(t,u)\,du\right) \tag{5}$$

and

$$r_r(t,\tau)=\frac{1}{\tau}\int_t^{t+\tau} f_r(t,u)\,du \tag{6}$$

When zero coupon prices are defined as functions of forward interest rates, the following developed representations hold for coupon-paying conventional and indexed bonds, respectively:

$$B_n(t,m)=\sum_{\tau=1}^{m} C\exp\left(-\int_t^{t+\tau} f_n(t,u)\,du\right)+VF\exp\left(-\int_t^{t+m} f_n(t,u)\,du\right) \tag{7}$$

and

$$B_{tips}(t,m)=IF_t\left[\sum_{\tau=1}^{m} C\exp\left(-\int_t^{t+\tau} f_r(t,u)\,du\right)+VF\exp\left(-\int_t^{t+m} f_r(t,u)\,du\right)\right] \tag{8}$$

The difference between the nominal and real spot rates $r_n(t,\tau)-r_r(t,\tau)$ is break-even inflation, or inflation compensation, composed of expected inflation during the periods between t and $t+\tau$ plus an inflation risk premium. Similarly, $f_n(t,u)-f_r(t,u)$ is the break-even inflation anticipated for time $t+u$.

The Jarrow and Yildirim method to estimate zero coupon bond prices relies on a discrete time approach to modeling forward interest rates and, additionally, it accepts the assumption that forward rates are constant within piecewise segments of the maturity spectrum. Under this method, the theoretical price function of a coupon bond, both conventional and indexed, has the following representation:

$$B(t,m)=\sum_{\tau=1}^{m} C_\tau \exp\left(-\left(\sum_{i=1}^{K} f_i\phi(\tau,i)\right)\right) \tag{9}$$

where C_τ is the bond payoff at date $t+\tau$, K is the number of piecewise maturity segments of constant forward rates, and f_i is the forward rate to be observed within the i^{th} maturity segment. Finally, $\phi(\tau,i)$ is the part of the i^{th} maturity segment that covers the C_τ payoff maturity, *i.e.*:

$\phi(\tau,i)=i-(i-1)$ if $\tau\geq i$,

$\phi(\tau,i)=\tau-(i-1)$ if $i>\tau\geq i-1$, and

$\phi(\tau,i)=0$ if $\tau<i-1$.

The bootstrapping method can be regarded as a particular case of the piecewise method employed by Jarrow and Yildirim (2003), in which the limits between two piecewise segments are set at the maturity dates of the coupon bonds, and the number of piecewise constant forward rates is equal to the number of coupon bonds in the sample. Estimation of the consecutive forward rates across the maturity segments, given by the bootstrapping procedure, produces estimated bond prices that match perfectly the market prices. Although this is a not significant advantage over the other methods, bootstrapping is much more machine- and time-consuming in computation than the other methods.

2.2 *Smoothing methods*

A- Cubic spline method

McCulloch (1975) fits a cubic spline to the discount function itself. A cubic spline discount function has the advantage of making the pricing equation for coupon bonds linear in unknown parameters that can be found with least squares. The cubic spline method addresses the first issue by dividing the term structure into a specific number of segments. Cubic functions are used to fit the term structure over these segments. The number of segments determines the number of points, called knot points, where the continuity and derivability of the various cubic functions are both assured as follows.

$$P(t,\tau) = 1 + \sum_{i=1}^{I=s} \propto_i g_i(\tau) \quad (10)$$

where s is number of segments, T_0 to T_s are knot points, α_j, i=1,….s are parameters to be estimated, and $g_i\ (\tau)$ is the i^{th} cubic function. The constraints of continuity and derivability applied to the cubic functions lead to more precise definition of the $g_i\,(\tau)$ functions, as follows:
Case 1: $i<s$

$g_i(\tau) = 0$ for $\tau \le T_{i-1}$

$g_i(\tau) = (\tau - T_{i-1})^3/6(T_i - T_{i-1})$ for $T_{i-1} \le \tau < T_i$

$g_i(\tau) = (T_i - T_{i-1})^2/6 + (T_i - T_{i-1})(\tau - T_i)/2 + (\tau - T_i)^2/2 - (t - T_i)^3/6(T_{i+1} - T_i)$ for $T_i \le \tau < T_{i+1}$

$g_i(\tau) = (T_{i+1} - T_{i-1})\ (2T_{i+1} - T_i - T_{i-1})/6 + (\tau - T_{i+1}/2)$ for $t \ge T_{i+1}$

Case 2: i=s, then $g_i(\tau) = \tau$

Substituting the characterization of $P(t,\tau)$ into the pricing formula of a coupon-bearing bond (equation (1) and (2)) allows estimation of the α_i parameters.

B- Nelson and Siegel model

Nelson and Siegel (1987) propose a parsimonious approach in modeling the forward curve using a function over the entire maturity range. These authors suggest a parameterization of the forward rate as an exponential function with four parameters: β_0, β_1, β_2, and η:

$$f(t,\tau)=\beta_0+\beta_1\exp\left(-\tau/\eta\right)+\beta_2\left[\tau/\eta\exp\left(-\tau/\eta\right)\right] \quad (11)$$

where $\beta_0 + \beta_1$ is the short-term rate and β_0 the consol rate. β_2 affects the curvature of the term structure and η is the speed of convergence of the term structure towards the consol rate.

Taking into consideration that the zero coupon rate is

$r(t,\tau)=\frac{1}{\tau}\int_t^{t+\tau} f(t,u)du$,

the following equation can be derived for the zero coupon bond:

$$P(t,\tau)=\exp\left[-\beta_0\tau-\eta(\beta_1+\beta_2)\left(1-\exp\left(-\tau/\eta\right)\right)+\beta_2\tau\exp\left(-\tau/\eta\right)\right] \quad (12)$$

Substituting this functional form into the pricing formula of a coupon-bearing bond (equations (1) and (2)), the four parameters can be estimated by minimizing the sum of squared errors between market and theoretical prices.

2.3 Hedging bond portfolios with inflation-indexed and nominal bonds

Hedging a bond portfolio consists of protecting it against nominal interest rate risk or, alternatively, against real interest rate risk and inflation risk. If the inflation compensation contained in the difference between nominal and real interest rates were a good forecast of future inflation, the two hedging solutions would be equivalent. These two hedging methods are tested.

A hedging strategy consists of constructing a portfolio with several different bonds whose proportions ensure that the portfolio has a zero derivative (or zero delta) relative to one or more risk factors. Inflation-linked bonds have non-zero deltas relative to real interest rates and inflation (represented by changes in the bond indexation factors), and zero deltas relative to nominal interest rates. Conversely, nominal bonds have non-zero deltas relative to nominal interest rates and zero deltas relative to real interest rates and to inflation. Frequently, nominal interest rates are correlated with real interest rates and inflation. Hence, both types of bonds depend directly or indirectly on three factors: nominal interest rates, real interest rates and inflation.

Let us consider a portfolio with zero initial value comprising one nominal bond (referred to hereafter as NOM) and three different TIPS. The zero deltas relative to real interest rates and to inflation are defined in the following system of equations:

$$\begin{aligned}
&1B_{tips1}(t)+q_1B_{nom}(t)+q_2B_{tips2}(t)+q_3B_{tips3}(t)=0\\
&\frac{\partial B_{tips1}(t)}{\partial r_r(t)}+q_1\frac{\partial B_{nom}(t)}{\partial r_r(t)}+q_2\frac{\partial B_{tips2}(t)}{\partial r_r(t)}+q_3\frac{\partial B_{tips3}(t)}{\partial r_r(t)}=0\\
&\frac{\partial B_{tips1}(t)}{\partial I(t)}+q_1\frac{\partial B_{nom}(t)}{\partial I(t)}+q_2\frac{\partial B_{tips2}(t)}{\partial I(t)}+q_3\frac{\partial B_{tips3}(t)}{\partial I(t)}=0
\end{aligned} \quad (13)$$

Under the market completeness assumption, the value of this portfolio being zero on date t should also be zero on any future date $t+\Delta$. The deviations from zero at future dates are hedging errors, which implies that the synthetic construction is imperfect. In Section 3 we implement each hedging strategy over the entire sample for different values of Δ.
The deltas of coupon bonds relative to the risk factors are provided by their payoff discount function deltas. Hence, the TIPS deltas relative to the real interest rate are given by the following expression:

$$\frac{\partial B_{tips}(t,m)}{\partial r_r(t)} = IF_t \left[\sum_{\tau=1}^{m} C \frac{\partial P_r(t,\tau)}{\partial r_r(t)} + VF \frac{\partial P_r(t,m)}{\partial r_r(t)} \right] \tag{14}$$

The TIPS payoffs present value deltas relative to the indexation factor are:

$$\frac{\partial \left[IF_t \left(CP_r(t,\tau) \right) \right]}{\partial IF_t} = CP_r(t,\tau) \tag{15}$$

Hence, the TIPS delta relative to the indexation factor is:

$$\frac{\partial B_{tips}(t,m)}{\partial IF_t} = \sum_{\tau=1}^{m} CP_r(t,\tau) + VFP_r(t,m) \tag{16}$$

Calculation of the TIPS deltas relative to the real interest rate depends on the chosen term structure model.
Jarrow and Yildirim (2003) developed a pricing model for TIPS that relies on a foreign currency analogy. These authors consider a hypothetical cross-currency economy under a no-arbitrage assumption where nominal currency corresponds to the domestic currency, real currency to the foreign currency, and the inflation index to the spot exchange rate. Hence, the model developed by Jarrow and Yildirim is a three-factor model.
Under the historical probability *P*, the three-factor model of Jarrow and Yildirim (2003) is defined by the dynamics of its factors.

The dynamic of the forward rate is:

$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dw_r(t) \;; \tag{17}$$

the dynamic of the nominal forward rate is:

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dw_n(t) \;; \tag{18}$$

and finally, the evolution of the inflation index, which allows the logarithm of the inflation index process to be normally distributed is:

$$dI(t)/I(t)=\mu_I(t)dt+\mu_I(t)dw_I(t) \qquad (19)$$

where $dw_r(t), dw_n(t)$ and $dw_I(t)$ are Brownian motions with the following correlations: $dw_n(t)dw_r(t)=\rho_{nr}dt$, $dw_r(t)dw_I(t)=\rho_{rI}dt$.

Under the Martingale Measure Q, the following processes hold, for nominal zero coupon bonds:

$$\frac{dP_n(t,T)}{P_n(t,T)}=r_n(t)dt-\int_t^T\sigma_n(t,s)dW_n(t) \; ; \qquad (20)$$

and for real zero coupon bonds:

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t)-\rho_{rI}\sigma_I(t)\int_t^T\sigma_r(t,s)ds\right]dt-\int_t^T\sigma_r(t,s)ds\,dw_r(t) \qquad (21)$$

where $r_n(t)$ and $r_r(t)$ are, respectively, the nominal and the real spot rate.

To make the model tractable, it is necessary only to specify $\sigma_r(t,T)$, the volatility function of the real forward rates and $\sigma_n(t,T)$, the volatility function of the nominal rates. We impose, as do Jarrow and Yildirim, an exponentially declining volatility for both functions:

$$\sigma_i(t,T)=\sigma_i e^{-\alpha_i(T-t)}(8) \text{ for } i=r \text{ (real)}, n \text{ (nominal)} \qquad (22)$$

which is an extension of the Vasicek (1977) model for the term structure, where σ_r et α_r et (respectively σ_n and α_n) are constants.
In this particular case, the bond return $dP_r(t,T)/P_r(t,T)$ follows a normal distribution characterized by a volatility parameter equals to $\int_t^T\sigma_r(t,s)ds$, which can be easily calculated if the $\sigma_r(t,T)$ *is* defined by equation (22):

$$\int_t^T\sigma_r(t,s)ds=\sigma_r\frac{\left(1-e^{-\alpha_r(T-t)}\right)}{\alpha_r} \qquad (23)$$

The same equation can be written for the nominal bond return $dP_n(t,T)/P_n(t,T)$:

$$\int_t^T\sigma_n(t,s)ds = \sigma_n\frac{\left(1-e^{-\alpha_n(T-t)}\right)}{\alpha_n} \qquad (24)$$

The volatility functions defined above are those of the extended Vasicek model. The Vasicek delta of the real zero coupon bond is:

$$\frac{\partial P_r(t,\tau)}{\partial r_r(t)}=\left(\frac{1-e^{-\alpha_r\tau}}{\alpha_r}\right)P_r(t,\tau) \qquad (25)$$

And for the nominal zero-coupon bond, it is:

$$\frac{\partial P_n(t,\tau)}{\partial r_n(t)} = \left(\frac{1-e^{-\alpha_n \tau}}{\alpha_n}\right) P_n(t,\tau) \quad (26)$$

where α_r (α_n) are the elasticities of reversion of the real (nominal) short-term interest rate to the long-term value. Therefore, The TIPS delta relative to the short-term real interest rate is:

$$\frac{\partial B_{tips}(t,m)}{\partial r_r(t)} = IF_t \left[\sum_{\tau=1}^{m} C \left(\frac{1-e^{-\alpha_r \tau}}{\alpha_r}\right) P_r(t,\tau) + VF \left(\frac{1-e^{-\alpha_r m}}{\alpha_r}\right) P_r(t,m) \right] \quad (27)$$

The Heath, Jarrow, Morton (1983) model was initially applied to nominal bonds. According to this model and under the assumption that the short-term nominal interest rate governs the nominal term structure, the Vasicek deltas of a nominal bond are given by the following expression:

$$\frac{\partial B_{nom}(t,m)}{\partial r_n(t)} = \sum_{\tau=1}^{m} C \left(\frac{1-e^{-\alpha_n \tau}}{\alpha_n}\right) P_n(t,\tau) + VF \left(\frac{1-e^{-\alpha_n m}}{\alpha_n}\right) P_n(t,m) \quad (28)$$

The Jarrow and Yildirim (2003) hedging ratios and the Fisher and Weil (1971) duration measure are used.

The Fisher and Weil duration is based on the assumption that stochastic changes in interest rates always consist of parallel shifts of the term structure. This duration measure has been used frequently in bond portfolio immunization strategies, whether as the single interest rate risk measure, as in Fong and Vasicek (1984), or with higher order interest rate risk measures, as in Nawalkha (1995) and Nawalkha, De Soto and Zhang (2003). The Fisher and Weil duration of a zero coupon bond is the sensitivity of its price relative to the corresponding spot rate, that is:

$$D_z(t,\tau) = -\frac{\partial P(t,\tau)}{\partial r(t,\tau)} \frac{1}{P(t,\tau)} \quad (29)$$

The equality $D_z(t,\tau) = \tau$ results from $P(t,\tau) = e^{-\tau r(t,\tau)}$, where $r(t,\tau)$ is the τ period maturity spot rate. A parallel shock on the term structure is represented with a stochastic variable $\phi(t)$, which satisfies the condition $\Delta r(t,\tau) = \phi(t)$ for $\tau=1,\ldots,m$. In this case, the derivatives of the zero coupon bond prices relative to the corresponding spot rates $\frac{\partial P(t,\tau)}{\partial r(t,\tau)} = -\tau P(t,\tau)$ can be replaced with the derivatives relative to the variable $\phi(t)$, $\frac{\partial P(t,\tau)}{\partial \phi(t)} = -\tau P(t,\tau)$. Under this approach, the TIPS delta relative to a parallel shock on the real term structure $\phi_r(t)$ is:

$$\frac{\partial B_{tips}(t,m)}{\partial \phi_r(t)} = -IF_t\left[\sum_{\tau=1}^{m} \tau CP_r(t,\tau) - mVFP_r(t,m)\right] \tag{30}$$

and the nominal bond delta relative to a parallel shock on the nominal term structure $\phi_n(t)$ is:

$$\frac{\partial B_{nom}(t,m)}{\partial \phi_n(t)} = -\sum_{\tau=1}^{m} \tau CP_n(t,\tau) - mVFP_n(t,m) \tag{31}$$

The use of alternative delta measures in the hedging strategies developed in this paper is similar to the use of alternative duration measures in the immunization literature; that is, it consists of determining whether the portfolio hedging performance differs significantly from one delta measure to another.

In addition, in the Vasicek deltas, different values of α_n and α_r are used to evaluate the dependence of a hedging strategy on the speed of convergence of interest rates to their long-term values.

Other portfolios comprising only nominal bonds are constituted and hedged against nominal interest rate risk. Such portfolios include three nominal bonds whose proportions are determined by the following system of equations:

$$\begin{aligned} &1B_{nom1}(t) + q_1 B_{nom2}(t) + q_2 B_{nom3}(t) = 0 \\ &\frac{\partial B_{nom1}(t)}{\partial r_n(t)} + q_1 \frac{\partial B_{nom2}(t)}{\partial r_n(t)} + q_2 \frac{\partial B_{nom3}(t)}{\partial r_n(t)} = 0 \end{aligned} \tag{32}$$

Comparing the hedging results of the two types of portfolios enables us to assess whether an investment in inflation-indexed bonds improves hedging efficiency, even in periods of low inflation rates.

3. Data description and presentation of results of pricing estimation and hedging strategy implementation

3.1. Data presentation and preliminary statistics

The database used in this paper for the implementation of bond pricing models and hedging strategies comprises daily data covering the period January 1, 2013 through December 31, 2015, or 783 daily market price observations and indexation factors of French inflation-linked OATs (Obligations Assimilables du Trésor) and prices of conventional OATs. Table I shows the list of the indexed and conventional bonds used in the paper, an and their issuance year, coupon, and maturity**.** In this table, the initials EI designate euro area inflation-linked bonds, while the initial I designates French inflation-linked bonds. Issuance year, coupon rate, and maturity date comprise the remaining data identifying the bonds. The following indexed bonds are excluded: OAT-EI 2014 0.7% 25/07/30 and OAT-I 2015 0.1% 01/03/25, because their prices are not available for the entire sample period, and OAT-EI 2002 3.15% 25/07/32 and OAT-EI 2007 1.8% 25/07/40, because their maturity dates are too remote from the other bonds in the sample.

Table 1: List of the indexed and conventional bonds used in the paper

Inflation-linked OAT: issuance year, coupon, and maturity	Conventional OAT: issuance year, coupon, and maturity
BTAN 2011 0.45% 25/07/16	BTAN 2011 2 1/2% 25/07/16
OAT-I 2005 1% 25/07/17	OAT 2007 4 1/4% 25/10/17
OAT-EI 2012 1/4% 25/07/18	OAT 2008 4 1/4% 25/10/18
OAT-I 2010 1.3% 25/07/19	OAT 1989 8 1/2% 25/10/19
OAT-EI 2004 2 1/4% 25/07/20	OAT 2010 2 1/2% 25/10/20
OAT-I 2012 0.1% 25/07/21	OAT 2011 3 1/4% 25/10/21
OAT-EI 2010 1.1% 25/07/22	OAT 1992 8 1/4% 25/04/22
OAT-I 2008 2.1% 25/07/23	OAT 1992 8 1/2% 25/04/23
OAT-EI 2013 1/4% 25/07/24	OAT 1994 6% 25/10/25
OAT-EI 2011 1.85% 25/07/27	OAT 2010 3 1/2% 25/04/26
OAT-I 1999 3.4% 25/07/29	OAT 2012 2 3/4% 25/10/27
	OAT 1998 5 1/2% 25/04/29

The period begins in 2013 because too few inflation-linked OATs were traded before that year.

For comparison purposes, we use a similar number of conventional OATs and inflation-linked OATs, even though there are many more conventional OATs.

A preliminary analysis of the coupon characteristics of the selected indexed and conventional bonds is presented in Table 2. The difference between the TIPS coupons in Table 2 and their coupon rates in Table 1 results from the effect of the indexation factor.

Table 2: Coupon characteristics of the sample of indexed and conventional OATs

Observation dates Bond maturity	Conventional	01/01/2013 Indexed	01/01/2014 Indexed	01/01/2015 Indexed
2016	2.5			
2017	4.25	0.468	0.470	0,472
2018	4.25	1.123	1.128	1.133
2019	8.5	0.258	0.259	0.260
2020	2.5	1.373	1.380	1.386
2021	3.25	2.721	2.739	2.748
2022	8.25	0.1	0.101	0.101
2023	8.5	1.183	1.190	1.194
2025	6	2.285	2.297	2.306
2026	3.5	NA	0.253	0.254
2027	2.75	1.96	1.973	1.979
2029	5.5	4.198	4.236	4.252
		0	0.000	0.000
MEAN	4.980	1.567	1.457	1.462
STDV	2.341	1.264	1.274	1.278
STDV/MEAN	0.470	0.807	0.874	0.874

As discussed above, the TIPS indexation factor is the ratio between the value of the harmonized CPI excluding tobacco I_t on date *t* and the value I_{t0} of the index on bond issuance date t_0. Since the harmonized CPI is published quarterly or monthly, the indexation factor value on each transaction date, *t,* is calculated with a linear interpolation between the CPIs at *t-m2* and *t-m3*, which are, respectively, the third and second months preceding the transaction date *t*. This method of indexation is applied to both French inflation-indexed bonds (OATIs) and euro area inflation-indexed bonds (OATEIs). The pairwise correlation coefficients between the variations of the OAT indexation factors are very high, not only within each group (OATIs/OATEIs), but also between the two groups. They are close to 1 for four OATIs on one side, four OATEIs on the other side and above 0.8 between the two groups. These are representative samples.

Table 3: Correlation coefficients among the indexation factor changes

	OEI12_18	*OEI04_20*	*OEI10_22*	*OEI13_24*	*OAI05_17*	*OATI10_19*	*OATI12_21*	*OAI08_23*
OEI12_18	1							
OEI04_20	0.999848	1						
OEI10_22	0.999845	0.999859	1					
OEI13_24	0.999805	0.99985	0.999821	1				
OAI05_17	0.820918	0.82084	0.820749	0.820575	1			
OATI10_19	0.820368	0.820282	0.820183	0.820008	0.999704	1		
OATI12_21	0.820933	0.820845	0.82076	0.820574	0.999683	0.999666	1	
OAI08_23	0.820436	0.820319	0.820251	0.820069	0.999664	0.999636	0.999628	1

3.2. Results of pricing models estimation

As mentioned in Section 2, four alternative pricing methods are used for TIPS and NOMs: 1) bootstrapping; 2) Jarrow and Yildirim piecewise constant forward rates; 3) McCulloch cubic spline; and 4) Nelson and Siegel forward rates exponential polynomial. The estimations result from the minimization of the sum of squared differences between market prices and prices given by the models that, in the case of TIPS, corresponds to:

$$\min \sum_{q=1}^{N} \left\{ B_{tips}\left(t, m_q\right) - IF_t \left[\sum_{\tau=1}^{m} CP_r\left(t, \tau_q\right) + VFP_r\left(t, m_q\right) \right] \right\}^2 \tag{33}$$

and in the case of the NOMs, corresponds to:

$$\min \sum_{q=1}^{N} \left\{ B_{nom}\left(t, m_q\right) - \left[\sum_{\tau=1}^{m_q} CP_r\left(t, \tau_q\right) + VFP_r\left(t, m_q\right) \right] \right\}^2 \tag{34}$$

where N is the number of bonds used in the estimation. To compare the ability of the implemented pricing procedures to reproduce market prices, the sum of the squared errors of each estimation is divided by the number of bonds used to compute the squared error per bond, *SQE*, which, for TIPS, is

$$SQE_{tips} = \frac{\left\{ B_{tips}\left(t, m\right) - IF_t \left[\sum_{\tau=1}^{m} CP_r\left(t, \tau\right) + VFP_r\left(t, m\right) \right] \right\}^2}{N} \tag{35}$$

and for NOMs is

$$SQE_{nom} = \frac{\left\{ B_{nom}\left(t, m\right) - \left[\sum_{\tau=1}^{m} CP_r\left(t, \tau\right) + VFP_r\left(t, m\right) \right] \right\}^2}{N} \tag{36}$$

The statistics (mean, standard deviation, maximum, and minimum) of SQE_{tips} and SQE_{nom} related to the four models estimated are presented, respectively, in Tables 4 and 5.

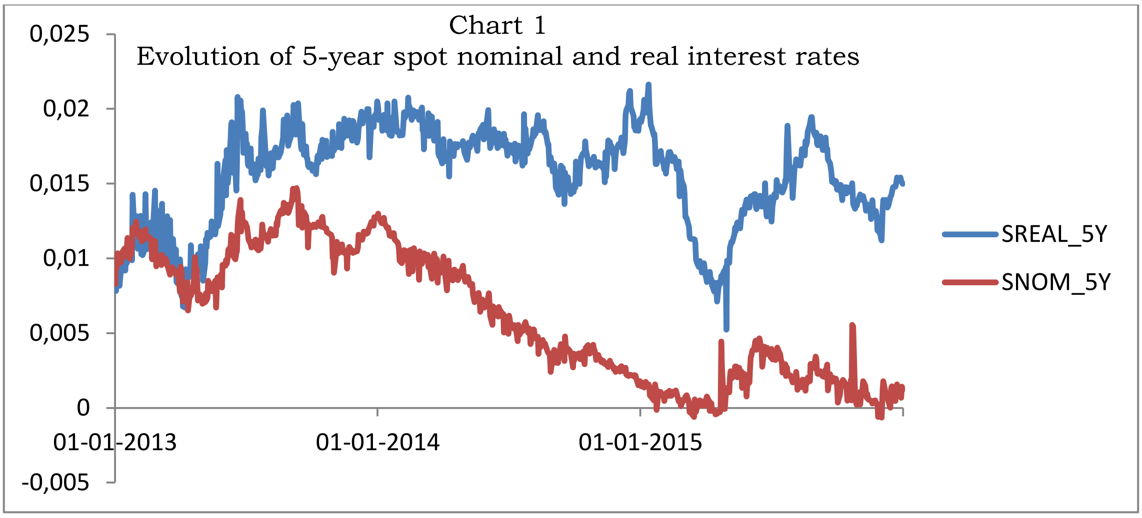
Table 4: Statistics of SQE_{tips}

	Bootstrapping	Jarrow & Yildirim	McCulloch	Nelson & Siegel
Mean	$2.63588\ 10^{-5}$	28.26182	29.10175	40.59776
St. Dev	0.000233141	1.213068	4.685037	8.864901
Max.	0.006450909	36.3141	122.4545	92.3217
Min	$4.78232\ 10^{-7}$	26.07986	26.3232	28.77388

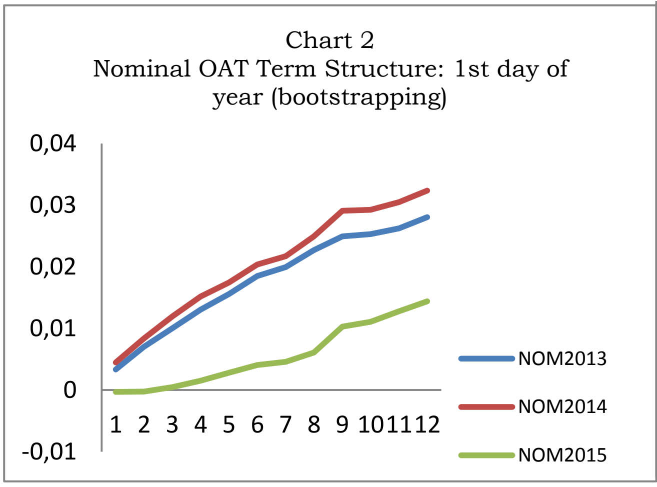
Table 5: Statistics of SQE_{nom}

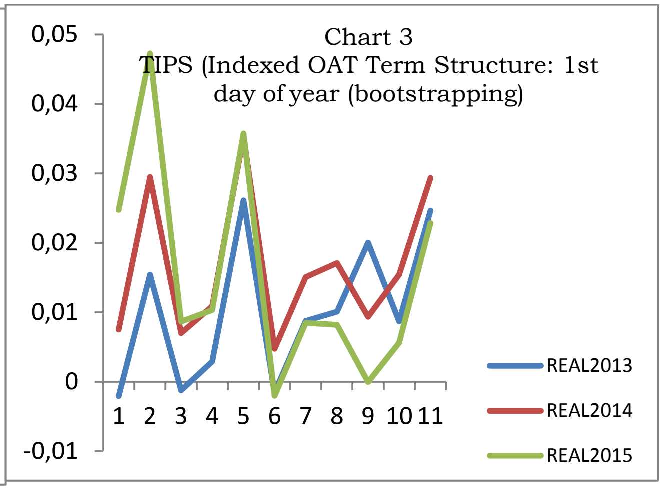
	Bootstrapping	Jarrow & Yildirim	McCulloch	Nelson & Siegel
Mean	$1.29826E\ 10^{-5}$	0.315437	0.304433	0.342886
St. Dev	$3.1909E\ 10^{-5}$	0.781258	0.978973	0.918475
Max.	0.000754167	7.767054	9.565738	10.14801
Min	$3.60719E\ 10^{-7}$	0.030788	0.002067	0.005123

The comparison between the SQE statistics presented in Tables 4 and 5 confirms that the bootstrapping method fits the market prices almost perfectly. These results also show that the other methods perform much better for nominal bonds than for indexed bonds and that the Nelson and Siegel method performs more poorly than the other methods for both bond groups. The difference between inflation-indexed bond market prices and the prices given by these models probably stems from the narrowness of the market in this bond segment, which comprises a limited number of bonds.
Chart 1, below, shows the evolution of the 5-year maturity spot nominal and real interest rates, estimated, respectively, with nominal OATs and indexed OATs. The paths followed by the two curves show that real interest rates extracted from TIPS prices are above nominal interest rates. Although inflation levels have been very low for the past several years, it does not seem reasonable to assume that expectations of negative inflation explain the negative difference between nominal interest rates and real interest rates. Indeed, the difference between the two curves may result from an abnormally high liquidity premium on TIPS, incurred from a loss of interest on the part of investors for these bonds. This may have pushed indexed bonds prices downwards and caused an exceptional increase in their relative yield.



Charts 2 through 9 graph the term structure provided by the different models for TIPS and NOMs on the first day of every year under study. The maturity of interest rates represented in Charts 2 and 3, given by bootstrapping, is the maturity of the coupon bonds in the sample. The other methods produce zero coupon bonds whose maturities are not limited to the maturities of the coupon bonds used in the estimations. Hence, Charts 4 through 9 represent zero coupon bond spot rates for yearly maturities ranging from 1 to 15 years.





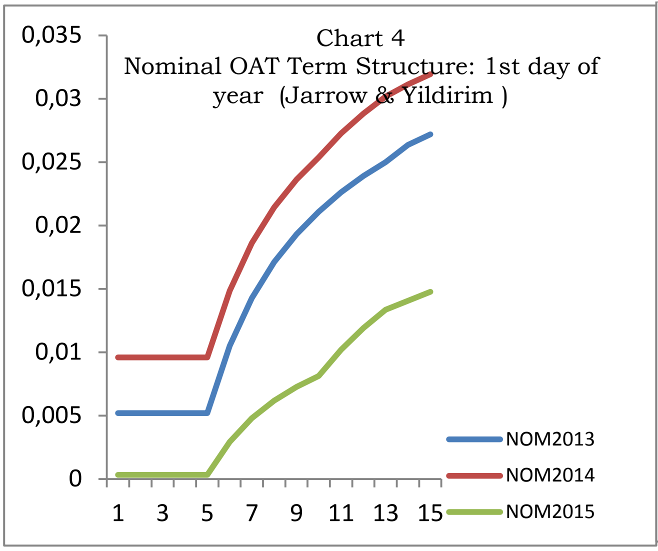


Chart 4
Nominal OAT Term Structure: 1st day of year (Jarrow & Yildirim)
0,035
0,03
0,025
0,02
0,015
0,01
0,005
0
1 3 5 7 9 11 13 15
NOM2013
NOM2014
NOM2015

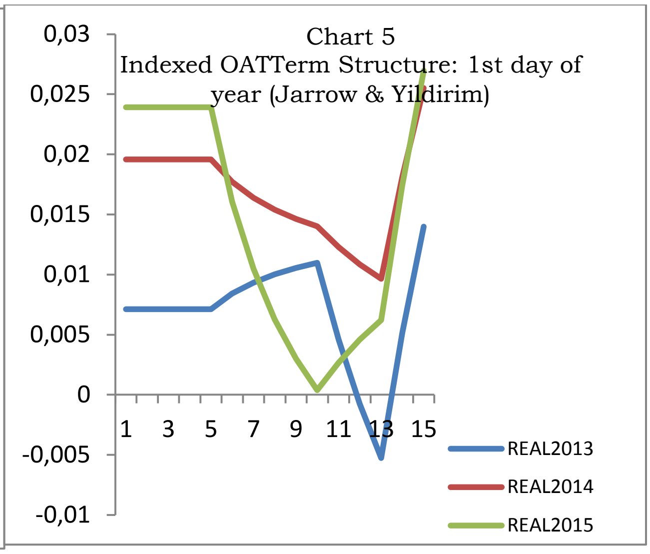


Chart 5
Indexed OATTerm Structure: 1st day of year (Jarrow & Yildirim)
0,03
0,025
0,02
0,015
0,01
0,005
0
-0,005
-0,01
1 3 5 7 9 11 13 15
REAL2013
REAL2014
REAL2015

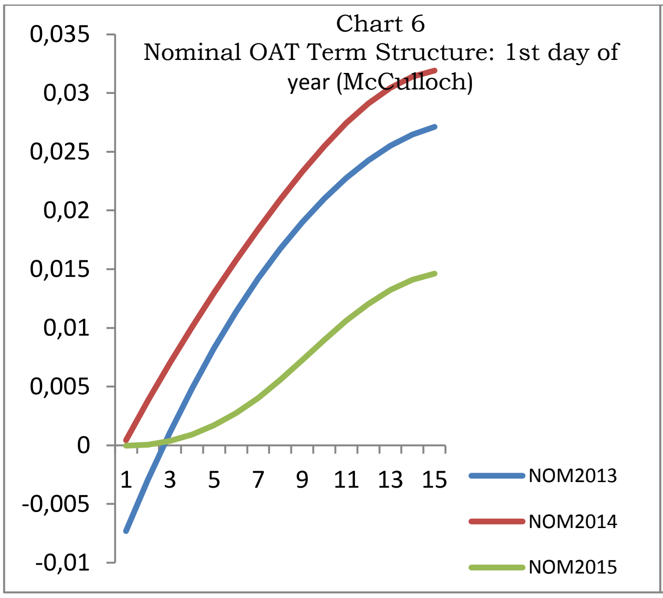


Chart 6
Nominal OAT Term Structure: 1st day of year (McCulloch)
0,035
0,03
0,025
0,02
0,015
0,01
0,005
0
-0,005
-0,01
1 3 5 7 9 11 13 15
NOM2013
NOM2014
NOM2015

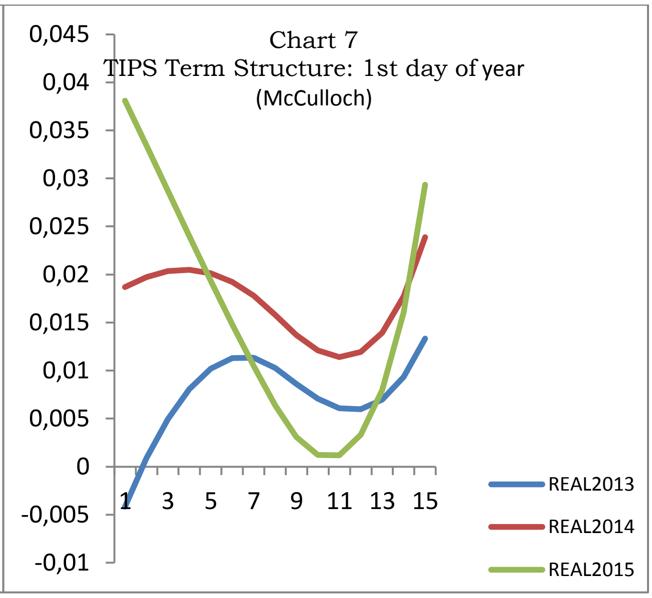


Chart 7
TIPS Term Structure: 1st day of year (McCulloch)
0,045
0,04
0,035
0,03
0,025
0,02
0,015
0,01
0,005
0
-0,005
-0,01
1 3 5 7 9 11 13 15
REAL2013
REAL2014
REAL2015

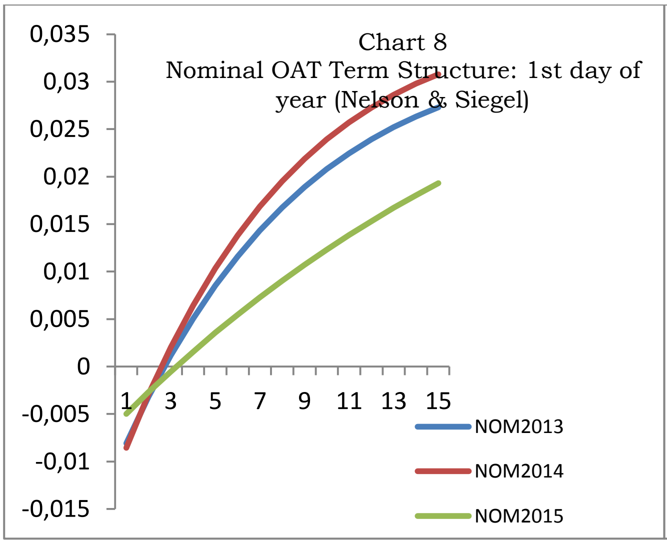


Chart 8
Nominal OAT Term Structure: 1st day of year (Nelson & Siegel)
0,035
0,03
0,025
0,02
0,015
0,01
0,005
0
-0,005
-0,01
-0,015
1 3 5 7 9 11 13 15
NOM2013
NOM2014
NOM2015

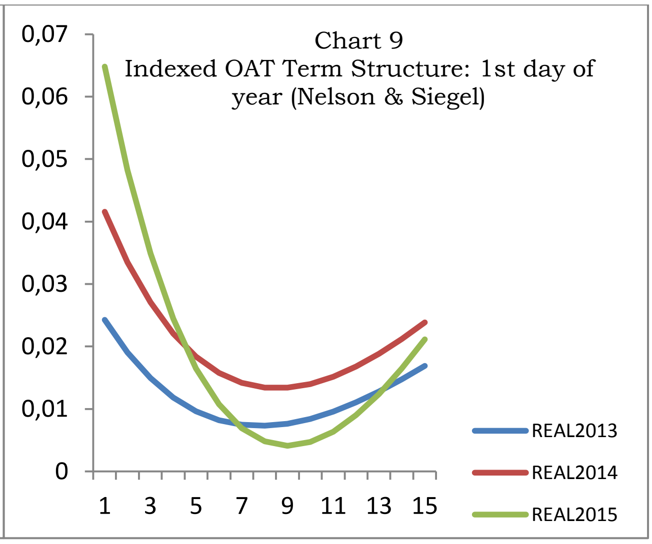


Chart 9
Indexed OAT Term Structure: 1st day of year (Nelson & Siegel)
0,07
0,06
0,05
0,04
0,03
0,02
0,01
0
1 3 5 7 9 11 13 15
REAL2013
REAL2014
REAL2015

Although the charts presented above evidence the changes in interest rate curves at one-year intervals, they cannot replace the information provided by the statistics on the zero coupon interest rates shown in Tables 6, 7, and 8.

Table 6. Statistics on zero coupon spot rates given by the piecewise forward Jarrow and Yildirim method

	Real Interest Rates				*Nominal Interest Rates*			
Maturity	*5 Years*	*7 Years*	*10 Years*	*15 Years*	*5 Years*	*7 Years*	*10 Years*	*15 Years*
Mean	0,017576	0,01016	0,004599	0,025551	0,003915	0,010903	0,016144	0,022553
St. Dev.	0,004442	0,004226	0,007745	0,012662	0,003834	0,005486	0,00686	0,007549
Kurtosis	0,410469	-0,14496	-1,29447	124,2216	-0,64055	-1,46532	-1,41558	-1,17615
Skewness	-0,92331	-0,49948	-0,29193	-6,06007	0,397301	-0,00514	-0,10033	-0,3115

Table 7: Statistics on zero coupon spot rates given by the McCulloch cubic spline method

	Real Interest Rates				*Nominal Interest Rates*			
	5 Years	*7 Years*	*10 Years*	*15 Years*	*5 Years*	*7 Years*	*10 Years*	*15 Years*
Mean	0,015515	0,010826	0,004519	0,026959	0,006292	0,010429	0,016544	0,022195
St. Dev.	0,003443	0,004937	0,005949	0,011897	0,004415	0,005713	0,006705	0,007868
Kurtosis	1,877199	-0,59038	-0,21661	2,634989	1,95E-05	3,26E-05	4,5E-05	6,19E-05
Skewness	-1,08427	-0,33413	-0,50631	1,419546	-1,40327	-1,46232	-1,3916	-1,28136

Table 8: Statistics on zero coupon spot rates given by the Nelson and Siegel method

	Real Interest Rates				*Nominal Interest Rates*			
	5 Years	*7 Years*	*10 Years*	*15 Years*	*5 Years*	*7 Years*	*10 Years*	*15 Years*
Mean	0,01342	0,007935	0,007699	0,020532	0,0069	0,01164	0,017187	0,023422
St. Dev.	0,003535	0,004834	0,005051	0,004512	0,004292	0,005438	0,006612	0,007535
Kurtosis	0,487765	0,25147	-0,38385	1,280516	1,84E-05	2,96E-05	4,37E-05	5,68E-05
Skewness	-0,86518	-0,62105	-0,27081	-0,5043	-1,31898	-1,35038	-1,23838	-0,97585

The statistics shown in Tables 6, 7, and 8 confirm, over the entire sample period, the monotonic increase in the nominal term structure, while the real term structure has a V shape, as illustrated by Charts 2 through 9. Kurtosis is also much higher for TIPS interest rates than for nominal OATs interest rates. Similar results are observed in Pericoli (2014) on French inflation-linked OATs, between 2004 and 2014, , which the author explains as resulting from the large segmentation and low liquidity of the euro area inflation-linked bond markets.

3.3 Hedging strategy results

Our empirical analysis covers bond portfolios composed of TIPS and NOMs on one hand, and bond portfolios composed of NOMs only on the other hand. As discussed in Section 2, the portfolio deltas relative to risk factors are based on two predominant models used in the literature to estimate bond interest rate risk: the hedge ratios of the Vasicek extended model and the Fisher and Weil duration.

According to the Vasicek model, the entire term structure is governed by the short-term interest rate, whose stochastic process is driven by the variable's elasticity of return to its long-term normal value—the parameter α represented in equations (23) through (25). The hedging strategies are implemented with three different values of α ($\alpha = 0.005$, $\alpha = 0.05$, and $\alpha = 0.1$). Different values for α enable us to determine the influence of the speed of the interest rate convergence on the results of the hedging strategies.

As discussed above, the Fisher and Weil duration is based on the assumption that the most frequent shocks to the term structure consist of parallel movements of the entire term structure.

Each hedged portfolio relies on one of the four rate extraction procedures and on the definition of the deltas, respectively, for the Vasicek model and the Fisher and Weil model. In addition, we use three holding periods (1, 5, and 10 days) to measure the impact of the holding period on hedging error. Thus, the hedged portfolios are constituted daily and reevaluated 1, 5, or 10 days thereafter.

The portfolios, composed of three TIPS and one NOM, are hedged against real interest rates and inflation risks based on the system of equations (19). They comprise one unit of OAT-I 2008 2.1% 25/07/23 and variable quantities of OAT-EI 2012 1/4% 25/07/18, OAT-I 2012 0.1% 25/07/21, and the NOM OAT 2010 2 1/2% 25/10/20. These bonds are chosen because their maturities, in the middle of the maturity spectrum, offer better protection for portfolio hedging than interest rates with longer terms. As Tables 6 through 8 show, longer term real interest rates are more leptokurtic and asymmetrically distributed than middle-term and short-term interest rates. Thus, they are more subject to abnormal changes than short- and mid-term interest rates.

From a theoretical point of view, French inflation and euro area inflation should be treated as separate factors, and an additional OATEI bond would be required to hedge these portfolios. However, the correlation between the two inflations is very high, as shown in Table 3. Thus, we assume that protection against French inflation ensures efficient protection against euro area inflation, and vice-versa.

The portfolios hedged against the nominal interest rate are composed of three nominal bonds with maturities in the middle-term spectrum, according to the system of equations (29): OAT 2008 4 1/4% 25/10/18, OAT 2010 2 1/2% 25/10/20, and OAT 1992 8 1/2% 25/04/23.

Table 9 recapitulates the means and standard deviations of the hedging errors (in percentage of bond nominal value as suggested by Jarrow and Yildirim (2003)). The results show that the pricing model and the delta definition do not play important roles in the efficiency of the hedging strategy. The same conclusion does not apply to the holding period, since hedging error increases with holding period length.

The statistics on the errors of hedging strategies on nominal interest rates show that results are similar for all pricing models and delta definitions. Results are dependent on the holding period. However, comparing the results shown in Table 9 with those of Table 10 reveals that combining indexed OAT with nominal OAT significantly reduces hedging errors compared to portfolios built with only conventional bonds. Thus, even in periods of low inflation, inflation-linked bonds enhance hedging efficiency. The most plausible explanation for this result is that indexedOAT and nominal OAT prices are weakly correlated when they do not vary in opposite directions.

Table 9: Hedged portfolios versus real interest rate and inflation risk

(Percentage Mean Hedging Error and Standard Deviation)

Hedging method	Period		Bootstrapping	J&Y	McCulloch	N&S
Vasicek α = 0.005	1 day	Mean	-0.000092	-0.000092	-0.000095	-0.000088
		St. Dev.	0.002892186	0.0030108	0.0030203	0.0029478
	5 days	Mean	-0.000471972	-0.000469	-0.000472	-0.000455
		St. Dev.	0.005024963	0.0052936	0.0053106	0.005134
	10 days	Mean	-0.000960575	-0.000959	-0.00097	-0.000928
		St. Dev.	0.006220463	0.0064214	0.0064686	0.0062637
Vasicek α = 0.05	1 day	Mean	-0.000089	-0.000089	-0.000091	-0.000088
		St. Dev.	0.002866974	0.0029728	0.0029818	0.0029478
	5 days	Mean	-0.000456732	-0.000453	-0.000455	-0.000455
		St. Dev.	0.00496109	0.0051887	0.005205	0.005134
	10 days	Mean	-0.000929572	-0.000926	-0.000935	-0.000928
		SDV	0.006156374	0.0063	0.0063454	0.0062637
Vasicek α = 0.1	1 day	Mean	-0.000087	-0.000092	-0.000089	-0.000085
		St. Dev.	0.002848034	0.003006	0.0029518	0.0029207
	5 days	Mean	-0.000443369	-0.000467	-0.000441	-0.000441
		St. Dev.	0.004913036	0.0052805	0.0051213	0.0050588
	10 days	Mean	-0.000902555	-0.000955	-0.000905	-0.000899
		St. Dev.	0.006112543	0.006406	0.0062517	0.0061814
Fisher Weil	1 day	Mean	-0.000092	-0.000092	-0.000095	-0.000091
		St. Dev.	0.002895523	0.0030157	0.0030253	0.0029875
	5 days	Mean	-0.000473853	-0.000471	-0.000474	-0.000473
		St. Dev.	0.005033431	0.0053072	0.0053243	0.0052425
	10 days	Mean	-0.000964409	-0.000963	-0.000974	-0.000964
		St. Dev.	0.006229282	0.0064374	0.0064848	0.0063881

Table 10: Hedged portfolios versus nominal interest rate risk
(Percentage Mean Hedging Error and Standard Deviation)

Hedging method	Period		Bootstrapping	J&Y	McCulloch	N&S
Vasicek α = 0.005	1 day	Mean	0.000106545	0.000102	0.00010279	0.000103
		St. Dev.	0.002443822	0.002377	0.002387659	0.002387
	5 days	Mean	0.000525646	0.000511	0.000519243	0.000523
		St. Dev.	0.00586232	0.005703	0.005728968	0.005736
	10 days	Mean	0.00106224	0.001033	0.001052865	0.001057
		St. Dev.	0.008357393	0.008141	0.008184737	0.008208
Vasicek α = 0.05	1 day	Mean	0.000107774	0.000101	0.000102879	0.000104
		St. Dev.	0.002663582	0.002578	0.002590307	0.00259
	5 days	Mean	0.000529445	0.000511	0.000520888	0.000525
		St. Dev.	0.006414174	0.006214	0.006242006	0.006251
	10 days	Mean	0.001070468	0.001034	0.001057607	0.001064
		St. Dev.	0.009147134	0.008875	0.008924521	0.008955
Vasicek α = 0.1	1 day	Mean	0.000110556	0.000102	0.00010375	0.000105
		St. Dev.	0.002943992	0.00283	0.002842253	0.002843
	5 days	Mean	0.000539849	0.000516	0.000527531	0.000533
		St. Dev.	0.007123129	0.006858	0.006887488	0.006899
	10 days	Mean	0.001092229	0.001045	0.001073145	0.001082
		St. Dev.	0.010158646	0.0098	0.009854625	0.009899
Fisher Weil	1 day	Mean	0.000106463	0.000102	0.000102811	0.000752
		St. Dev.	0.002420964	0.002355	0.002366354	0.056093
	5 days	Mean	0.000525463	0.000511	0.00051925	-0.00699
		St. Dev.	0.005805075	0.00565	0.005675279	0.141458
	10 days	Mean	0.001061816	0.001033	0.001052745	-0.01059
		St. Dev.	0.008275347	0.008064	0.008107274	0.151334

4. Conclusions

This paper focuses on French Treasury inflation indexed bonds and their power to avoid the risk of inflation. In a first step, both the real and nominal zero-coupon yield curves are extracted from the market prices of French indexed and conventional bonds. Several methodologies are used to extract these zero coupon yields. In a second step, two hedging strategies are tested on two types of portfolio: a portfolio comprising indexed and conventional bonds and a portfolio comprising only conventional bonds.
Regarding the methodology of extraction, this paper highlights the power of adjustment of the bootstrapping method to the market prices of both indexed and conventional bonds. The adjustment between estimated prices and market prices given by the other methods described in this paper is better for conventional bonds than for

inflation indexed bonds. This is particularly the case for the Nelson and Siegel smoothing model. The difficulty in capturing the real interest rates curve could be due to a lack of efficiency of the price of these bonds given that these securities are probably less attractive for investors since inflation is low. In this case, these securities are probably less traded and could be subject to risk premium.

Regarding hedging strategies, we may conclude that the extraction method does not impact hedging results. The same conclusion applies for the methodology of hedging (Vasicek or Fisher and Weil), since hedging error does not differ from one methodology to another. On the other hand, the holding period has a great impact. However, accounting for inflation-linked bonds in a bond portfolio comprising conventional bonds significantly improves bond portfolio protection, even in a context of low inflation.

This result is of great interest for asset managers who intend to hedge their conventional bond portfolio against the risk of inflation.

From a theoretical point of view, we may conclude that the model of Jarrow and Yildirim (2003), which enables pricing inflation-linked bonds and hedging bond portfolios, relies on few parameters. However, this model necessarily requires extraction of the real and nominal zero coupon rates to be operational.

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