

Is Normal Backwardation Normal? Valuing Financial Futures with a Stochastic, Endogenous Index-Rate Covariance

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Abstract

Revisiting the two-factor valuation of financial futures contracts and their derivatives, we propose a new approach in which the covariance process between the underlying asset price and the money market interest rate is set endogenously according to investors' arbitrage operations. The asset-rate covariance turns out to be stochastic, thereby explicitly capturing futures contracts' marking-to-market feature. Our numerical simulations show significant deviations from the traditional cost-of-carry model of futures prices, in line with Cox, Ingersoll and Ross's (1981) theory and a large corpus of past empirical research. Our empirical tests show an impact of several index points magnitude from the recent US Federal Reserve interest rate hikes on the S&P 500 daily spot-futures basis, highlighting the effect of monetary policy at low frequencies on the backwardation vs. contango regime, and shedding new light on Keynes's (1930) theory of normal backwardation.

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Two economic explanations are usually put forward to account for the deviation observed between contemporaneous spot and futures prices, also called the spot-futures basis. Following Keynes's analysis (1930), the cost-of-carry hypothesis relies on the theory of storage (Kaldor, 1939; Working, 1948, 1949) to interpret the spot-futures basis in relation to several technical factors such as storage costs, convenience yields granted by immediate ownership of commodities, and interests forgone by storing commodities. Noticing that "under a regime of very widely fluctuating prices, the cost of insurance against price changes . . . is very high" (*A Treatise on Money*, p.142), Keynes adds another explanation: speculators will claim a premium in exchange for bearing the risk of price fluctuations. This fundamental insight originates the normal backwardation theory of the "forward market," in which the spot-futures basis arises from the superior forecast power of futures prices. Nevertheless, critical empirical studies (e.g., Fama and French, 1987; Kolb, 1992) have shown that the forecastability pattern of futures prices remains controversial, at the very least.¹

In this article, we explore another rationale that provides a competing explanation for the spot-futures basis of such financial assets as interest-earning assets and stock indexes. Since the seminal works of Black (1976), Jarrow and Oldfield (1981) and Cox, Ingersoll, and Ross (1981), we know that the value of a forward contract is generally not the same as that of a futures contract. Since its costs of marking to market induce regular repayments between the owner and the seller, a futures contract closely depends, until it matures, on the prevailing money market funding rate. The forward and futures prices consequently diverge as soon as interest rates become stochastic, and the difference depends on the covariance between the futures prices and the money market account, seen as the natural pricing numeraire of futures contracts. This article revisits the marking-to-market hypothesis, which posits that the market's expectations of the asset-rate covariance provide the main driver for the forward-futures price difference.

Although the local covariance between the underlying asset price and the short-term financing interest rate is generally supposed to be constant over time or to be deterministic, it is well-known

¹See Chow, McAleer, and Sequeira (2000) for an extensive survey of the futures pricing literature.

that this assumption is not at all realistic (e.g., French, 1983). In this study, we use two random processes to take into account the specificity of futures contracts: the first relates to the dynamics of the underlying asset price, and the second has to do with the evolution of the interest rate term structure. As a characteristic feature of our model, the covariance is mainly characterized by (i) being determined endogenously, thanks to the investors' arbitrage operations, and (ii) being variable over time, state-dependent and stochastic. To the best of our knowledge, this article is the first to endogenize asset-rate covariances in an arbitrage-free setting, thereby avoiding the theoretical apparatus of a general equilibrium framework.

Such a valuation setting sheds new light on the normal backwardation regime. The index-rate covariance being stochastic and determined by investors' operations, it moves along time and may become negative. In such a situation, an increase in interest rates results in a decrease in both the spot price and the expected future spot price, as well as an increase in the futures price, as the cost of carry becomes higher. Due to the conjunction of these two movements, the futures price may cross over the expected future spot price and establish a contango.

Our findings are of two kinds: theoretical and empirical. First, by introducing a stochastic endogenous covariance, our numerical simulations indicate significant deviations larger than 1% from the traditional cost-of-carry model of futures prices. Our model is thus able to capture a broad spectrum of configurations for the spot-futures basis. This flexibility leads to reassess the role of the interest rate term structure in the formation of financial futures prices with regard to the extant theoretical literature. Second, our empirical tests confirm the role of the market's interest rate expectations in the formation of financial futures risk premia at low frequencies. By using instrumental variables driven by the monetary policy expectations, we show that a proxy for the index-rate covariance is the preferred channel of transmission of the market's interest rate anticipations on the S&P 500 daily spot-futures basis. We are thus in a position to isolate the pure rate effect stemming from the recent interest rate hikes triggered by the US Federal Reserve. We find a deflating impact on the S&P 500 futures risk premium ranging from 2 index points (December 2016, June 2017) to 17 points (March 2017). Our findings stand in contrast to the contemporary

empirical literature on the S&P 500 basis which mainly focuses on its intraday dynamics.²

The article proceeds as follows. Section 1 reviews the futures literature. Section 2 puts forward the microeconomic foundations of the model in a discrete-time setting. Section 3 builds the endogenous stochastic covariance and derives our multinomial hybrid lattice. In Section 4, we conduct an extensive sensitivity analysis of financial futures contracts. In Section 5, an empirical analysis of the model is carried out on daily S&P 500 and Euro Stoxx 50 data which highlights the impact of monetary policy on the spot-futures basis. Finally, Section 6 concludes the article.

1. Literature Review

A forward contract commits the buyer to purchase the asset at the maturity date of the contract at a pre-agreed price. Worth zero at inception, such a contract incurs no intermediate cash-flows and thus admits the zero-coupon discount bond of the same maturity for natural pricing numeraire. By contrast, futures contracts are marked to the market on a daily basis and are generally resettled with respect to the underlying asset price (Black, 1976). This “marking-to-market” feature turns the money market account into the natural pricing numeraire for futures contracts. As recognized by Cox, Ingersoll, and Ross (1981) and Jarrow and Oldfield (1981), the daily reinvestment of futures’ steady streams of marked-to-market cash flows entails the divergence between futures prices and forward prices in the presence of interest rate uncertainty. Interest rates and the underlying asset price are thus the main risk factors necessary to capture the dynamics of financial futures prices (e.g., stock index futures). In the context of commodity derivatives, the convenience yield granted by immediate ownership of the physical commodity is usually taken into account in the modeling of futures as a third risk factor (Gibson and Schwartz, 1990). In the context of financial futures, this yield of convenience naturally translates into a stochastic dividend yield.

Empirical research finds support for the marking-to-market hypothesis or CIR effect (for Cox, Ingersoll and Ross) that the forward-futures price difference is driven by the daily resettlement fea-

²An extensive literature has developed that studies the intraday dynamics of the S&P 500 index-futures basis. See, for example, Kawaller, Koch, and Koch (1987), MacKinlay and Ramaswamy (1988), Stoll and Whaley (1990), Chan (1992), Miller, Muthuswamy, and Whaley (1994), Dwyer, Locke and Lu (1996), Roll, Schwartz, and Subrahmanyam (2007).

ture of futures contracts (e.g., Park and Chen, 1985). If French (1983) underscores the difficulties in estimating the market's expectation of local asset-rate covariances and provides limited empirical support for the CIR effect in commodity markets (copper and silver), significant CIR effects have been highlighted in foreign exchange markets (Cornell and Reinganum, 1981; Dezhbakhsh, 1994). The evidence appears more mixed in the case of stock index futures markets (MacKinlay and Ramaswamy, 1988), in which the impact of the marking-to-market feature may sometimes appear negligible in comparison with market imperfections such as transaction costs, bid-ask spreads, or market impact. By contrast, Sundaresan (1991), Meulbroek (1992) and Grinblatt and Jegadeesh (1996) find significant empirical support in the case of more interest-rate-sensitive financial assets, such as Eurodollar futures.

The empirical literature on the CIR effect belongs to a wider strand of the futures literature that strives to test Keynes's (1930) theory of normal backwardation—the property of the futures prices to evolve below the expected future spot price because of a risk premium. In the perfect-markets approach, early studies (e.g., Dusak, 1973; Carter, Rausser, and Schmitz, 1983; Chang, 1985; Fama and French, 1987; Kolb, 1992) find conflicting evidence of risk premia in commodity futures markets due to the absence of systematic risk. The alternative hypothesis, of imperfect markets, has consequently led researchers to reconsider the prevalent role of hedgers in the formation of futures risk premia. In this regard—and in contrast with commodity futures—financial futures do incur both systematic risk and hedging pressure (Bessembinder, 1992). In a study with results closely related to this article, De Roon, Nijman, and Veld (2000) provide evidence that financial futures risk premia are also conditioned by hedging pressures arising in other futures markets, such as interest rate markets.

From the theoretical point of view, scholars have proposed several two- or three-factor models for the valuation of futures contracts. Ramaswamy and Sundaresan's (1985) two-factor model assumes stochastic interest rates within the Cox, Ingersoll, and Ross (1985) general equilibrium framework in order to value American-style options on futures. However, the instantaneous correlation between the asset price and the short-term interest rate remains constant. Schwartz

(1997) is the first to introduce a three-factor model of futures contracts by assuming a simple mean-reverting process for the short-term, risk-free interest rate. He explicitly derives analytical forward and futures pricing formulae. Miltersen and Schwartz's (1998) three-factor model nests Schwartz's model in the multifactor, non-Markov interest rate framework of Heath, Jarrow, and Morton (1992). It is still possible to obtain closed-form solutions for forward and futures prices in their framework. Possible correlation among the three sources of risk, however, arises only through a common Wiener process, as the three diffusion parameters remain deterministic functions of the time parameter and do not allow for state-dependent correlation. In a similar article, Hilliard and Reis (1998) generalize the underlying asset price dynamics to the case of a jump diffusion. Their three-factor model—as well as the more recent ones by Casassus and Collin-Dufresne (2005) and Liu and Tang (2010)—still enable us to solve analytically for the futures price. They use arbitrage-free interest rate diffusions, making possible exact fits to the initial term structure of interest rates, but still rely on constant instantaneous correlation structures.

The Gaussian models of futures contracts developed so far view asset spot prices and interest rates as separate stochastic processes with a constant exogenous correlation. More recently, explicit stochastic covariance structures have been introduced in relation with commodity futures pricing (Chiu, Wong, and Zhao, 2015). However, the introduction of a continuous-time process for the stochastic covariance matrix between several asset spot price processes ignores the specificity of the asset-rate covariance. Moreover, the free-arbitrage setting of such models makes it difficult to endogenize covariances.

In parallel to a no-arbitrage approach to futures modeling, other authors have explored a general equilibrium approach resting upon economic primitives that allow for endogenous correlation between risk sources. Hemler and Longstaff (1991) develop a closed-form general equilibrium model of stock index futures prices with stochastic interest rates. Routledge, Seppi, and Spatt (2000) develop an equilibrium model of the term structure of forward prices for storable commodities, in which convenience yields and attached correlation structures arise endogenously.

2. Model Assumptions

In this section, we draw on the principles of analysis underscored in the previous section to build a set of representative assumptions to model financial futures contracts and the derivatives written on those assets, such as futures options.

2.1 A two-factor model

Generally speaking, the link between the evolution of the interest rate term structure and the value of a financial asset remains unclear. To address this problem, we use an arbitrage model whose main characteristic is to assume a stochastic covariance between asset prices and risk-free discount bond prices. The aim is to set up an underlying asset price process that depends, notably, on the evolution of the money market interest rate term structure.

ASSUMPTION 1 (Underlying asset price risk factors). *The underlying asset price process depends on:*

1. *the changes in the underlying asset's economic factors;*
2. *the changes in the cost of money reflected by the changes in the money market interest rate, which are equivalent to the funding costs incurred by the holder of a futures contract on her margin account.*

When the marking-to-market funding costs remain constant, the underlying asset's economic factors are the only ones that influence the underlying asset price behavior. The underlying asset price is then assumed to follow a classical Ito process, that is, a Boyle trinomial tree (1986, 1988) within a discrete time setting, which leads to a valuation model of contingent claims identical to those obtained by the Black and Scholes model (1973). Consequently, when affected by the combined influence of economic factors and money market interest rates, the underlying asset price evolves according to a multinomial lattice.

ASSUMPTION 2 (Multinomial lattice convergence). *When the money market interest rate stays at its initial level, the underlying asset price evolves according to its economic factors, and the multinomial lattice joins the classical trinomial model specified for the marginal evolution of the underlying asset.*

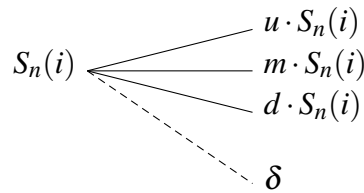
2.2 Marginal evolution of the underlying asset

ASSUMPTION 3 (Underlying asset price defaultable process). *In the absence of interest rate uncertainty, the underlying asset price is assumed to follow the subsequent risk-neutral process in a continuous-time setting:*

$$dS_t = (r_t - q_t + \lambda)S_t dt + \sigma S_t dW_t - S_t dN_t, \quad (1)$$

where r_t is the (risk-free) interest rate, q_t is the asset continuous dividend yield, σ is the asset volatility, W_t is a Wiener process and N_t is a Poisson process with intensity λ independent from W_t . In case of default ($dN_t = 1$), the asset price is supposed to drop to an absorbing default state δ .

We will discretize the underlying asset price defaultable process according to a quadrinomial lattice. The starting point S_0 of the lattice corresponds to $n = i = 0$, where n is the time index and the i index indicates that the bullish evolutions since the beginning of the lattice exceed the bearish ones by a number equal to i (a negative number means the bearish evolutions exceed the bullish ones). For some subsequent node $(n; i)$ the extended kernel is as follows:



where u (resp. m , d) is the upward (resp. stable, downward) transition multiplier. Of course, when the asset price trajectory drops down to the default state δ , it does not move further. The evolution

parameters are supposed to be constant over time and equal to the following values:

$$\begin{cases} u = e^{\theta\sigma\sqrt{\Delta t}}, \\ m = 1, \\ d = e^{-\theta\sigma\sqrt{\Delta t}}, \end{cases} \quad (2)$$

where the “stretch” parameter θ is greater than 1 (Boyle, 1988) to ensure consistent transition probabilities. The default probability q_0 is constant at each step of the process. Its value is given as follows:

$$q_0 = 1 - e^{-\lambda\Delta t}. \quad (3)$$

2.3 Margin account valuation

We take cognizance of the money market funding cost effective on the margin account of a futures contract holder using a generalized Vasicek model (1977). Note that there is no jump in this short rate process that could reflect any harsh modification in the funding cost, as a jump-to-default process has already been superposed upon the underlying asset price process.

ASSUMPTION 4 (Money market funding cost process). *The money market funding cost is supposed to follow the process described by the following stochastic dynamics:*

$$dr_t = (h_t - ar_t) dt + vdz_t, \quad (4)$$

where h_t is a drift term, a is the speed of the mean reversion, v is the interest rate volatility and z_t is a Wiener process driving term structure movements.

The stochastic dynamics for the short-term interest rate are chosen to be normal for tractability and expositional purposes.³ Hull and White (1990b, 1993) made the normal short-term rate process discrete, according to a trinomial tree. In a discrete world, the drift of the mean-reverting process

³Although it is still possible to use a log-normal interest rate diffusion (Black and Karasinski, 1991) to model short-term interest rates, a normal diffusion seems more realistic for the purpose of futures pricing in view of central banks’ recent monetary policies leading to negative overnight rates.

is formulated at node $(n; j)$ as $h(n\Delta t) - ar_j$, where n stands for time and j indexes the interest rate level. This arbitrage-free process becomes evident as soon as $h_n := h(n\Delta t)$ is set through an exact fit to the term structure. Considering the dynamics of the mean-reverting process and the yield to maturity of zero-coupon bonds, Hull and White (1993) find that:

$$h_n = (n+2)R_{n+2} + \frac{v^2\Delta t}{2} + \frac{1}{\Delta t^2} \ln \sum_j Q_n^j e^{-2r_j\Delta t + ar_j\Delta t^2}, \quad (5)$$

where R_{n+2} represents the yield (per period) of a zero-coupon bond reaching maturity in $n+2$, and Q_n^j is the $(n; j)$ -th Arrow-Debreu security price, i.e. the value at time zero of a security paying a monetary unit if the node $(n; j)$ is reached. Given the value of h_n and the interest rate step Δr being chosen, the move from node $(n; j)$ to node $(n+1; k)$ is done by choosing the index k , such that the median branch of the lattice at $n+1$ reaches a value r_k which is as close as possible to $r_j + h_n - ar_j$. The two other values for the short rate r at step $n+1$ may be deduced from r_k respectively by adding and subtracting the tick size Δr .

Let $p_n^{j,k-1}$, $p_n^{j,k}$ and $p_n^{j,k+1}$ be the risk-neutral probabilities of movements to nodes $(n+1; k-1)$, $(n+1; k)$ and $(n+1; k+1)$. If movements within the interest rate lattice are to respect moments of order 1 and order 2 of the short rate process, these probabilities are as follows:

$$\begin{cases} p_n^{j,k+1} = \frac{v^2\Delta t}{2\Delta r^2} + \frac{\eta^2}{2\Delta r^2} + \frac{\eta}{2\Delta r}, \\ p_n^{j,k} = 1 - \frac{v^2\Delta t}{\Delta r^2} - \frac{\eta^2}{\Delta r^2}, \\ p_n^{j,k-1} = \frac{v^2\Delta t}{2\Delta r^2} + \frac{\eta^2}{2\Delta r^2} - \frac{\eta}{2\Delta r}, \end{cases} \quad (6)$$

where $\eta := (j-k)\Delta r + (h_n - ar_j)\Delta t$. These probabilities must be positive or nil; this condition leads to a range of acceptable values for the interest-rate tick Δr .

2.4 A multinomial recombining hybrid lattice

The next assumption comes from the observation that options on financial futures are usually structured as American-style derivatives. Indeed, due to the very nature of futures contracts, which

are usually closed out before reaching maturity, derivatives on futures are inherently subject to premature exercise. As a result, it is highly desirable that the underlying asset price process be a Markov process that can be represented by a recombining lattice where the number of nodes grows linearly with the time variable.

ASSUMPTION 5 (Hybrid lattice recombination). *Any node inside the multinomial lattice may be reached by at least two different paths.*

2.5 Introducing the asset-rate covariance

To fully describe the evolution of the underlying asset price, we need to generalize the underlying asset's standard transition multipliers to a hybrid evolution.

Definition 1 (Hybrid transition multipliers). *In addition to the standard transition multipliers, the following hybrid transition multipliers are introduced:*

- $u_n^{j,k}$ is the evolution coefficient of the underlying asset price between time steps n and $n + 1$, when a rise in the asset economic factors occurs and when the margin account funding cost reaches level j at time step n and will attain level k at time step $n + 1$;
- $d_n^{j,k}$ is the evolution coefficient of the underlying asset price between time steps n and $n + 1$, when a fall in the asset economic factors occurs and when the margin account funding cost attains level j at time step n and will attain level k at time step $n + 1$.

Both evolution parameters have a multiplying effect on the asset price. However, when the margin account funding cost stays at its initial level (i.e., at $j = 0$), its recombining feature (see Assumption 5) ensures that the asset price's hybrid lattice joins the initial quadrinomial lattice.

The following values are then fixed:

$$\begin{cases} u_n^{0,0} = u, \\ d_n^{0,0} = u^{-1}. \end{cases} \quad (7)$$

The standard transition multipliers u and d may thus be viewed as hybrid transition multipliers of the underlying asset price when there is a bullish movement (or a bearish one) related to the economic factors and when interest rates remain at their initial level.

We now introduce a new asset-rate comovement factor designed to quantify the sensitivity of underlying asset prices to the changes in the margin account's funding cost. Following tradition, we shall refer to the term *contango*, which was commonly used on the London Stock Exchange until the 1930s to designate the fee that a buyer had to pay to a seller in order to defer the settlement of a trade.⁴

Definition 2 (Contango factor). *When the interest rate moves from level 0 to level k and everything else remains constant, at any time-step n the upward hybrid transition multiplier is written as:*

$$u_n^{0,k} = u \cdot \Phi_n^{0,k} \cdot \frac{1 + r_0}{1 + r_k}, \quad (8)$$

where $\Phi_n^{0,k}$ is the contango factor. Assuming that the modifications are identical whether the scenario proves to be bullish or bearish, at any time-step n the downward hybrid transition multiplier is written as:

$$d_n^{0,k} = d \cdot \Phi_n^{0,k} \cdot \frac{1 + r_0}{1 + r_k}. \quad (9)$$

To fully justify the “contango” terminology, we need a technical result linking the contango factor to the asset-rate covariance (see Section 3.4, Proposition 5). It is still possible at this stage, however, to provide a heuristic argument to understand why the futures price should be an increasing function of the contango factor. Let us suppose that the latter is greater than one. As upward moves of the interest rate produce depressing effects on the underlying asset price, such effects should be less accentuated than they would be with a neutral contango factor, say equal to one. As a result, the margin account of the holder of a long futures contract should tend to be more credited when interest rates are high, and less debited when interest rates are low. All else being equal, the

⁴First recorded in the mid 1800s in England, the term is considered to be an alteration of either the word *continuation*, the word *continue*, or the word *contingent*.

futures price should therefore be higher than it would be with a contango factor of less than one, leading in practice to a contango regime in which the futures price will evolve above the expected future spot price. Conversely, a contango factor of less than one should lead in practice to a regime of normal backwardation.

Note that Equations (8) and (9) are valid for any interest rate level k (positive, negative or zero). In particular, in case the interest rate remains constant (i.e., $k = 0$) we have $\Phi_n^{0,0} = 1$. More generally, the recombining feature of the hybrid lattice (see Assumption 5) ensures that for any index k :

$$\Phi_n^{k,k} = 1. \quad (10)$$

Each contango factor $\Phi_n^{0,k}$ is specific to an interest rate scenario $(n;k)$. No assumption is made either about the values of these parameters or about the relation that could exist between them. They may be, notably, superior or inferior to unity and therefore dampen or accelerate the movement caused by the variation of the interest rate. For each time-step n , there is a specific vector $\{\Phi_n^{0,k}\}_k$ which includes j_n data, where j_n is the number of interest rate scenarios specific to the time-step n .

Later, we shall show that when all the contango factors $\Phi_n^{0,k}$ are known, it becomes possible to calculate the hybrid transition multipliers $u_n^{j,k}$ and $d_n^{j,k}$ between any interest rate levels j and k . For that purpose, we will use the following generalization of Equation (10) at any interest rate level k :

$$\Phi_n^{k,k} = \Phi_n^{k,k+1} \cdot \Phi_n^{k+1,k} = 1, \quad (11)$$

as well as the following chain rules:

$$\Phi_n^{0,k+1} = \Phi_n^{0,k} \cdot \Phi_n^{k,k+1}, \quad (12)$$

$$\Phi_n^{0,k-1} = \Phi_n^{0,k} \cdot \Phi_n^{k,k-1}, \quad (13)$$

which are directly derived from definitions (8) and (9).

It is now possible to encapsulate the asset-rate sensitivity into a new auxiliary process, which will prove to be key when it comes to generalizing our discrete-time model in a continuous-time setting (see Appendix D).

Definition 3 (Contango process). *The contango process ρ_n^k is defined from the interest rate process and the contango factor as follows:*

$$\rho_n^k := \Phi_n^{0,k} \cdot \frac{1 + r_0}{1 + r_k}. \quad (14)$$

The amplitude of the contango process captures the propensity of the underlying asset price to co-evolve with the margin account funding cost.

At each node $(n; i, k)$ of the underlying asset price lattice, the bullish evolutions since the beginning of the hybrid lattice exceed the bearish ones by a number equal to i . If the interest rate level is indicated by k , the underlying asset price will now be written in the following compact form:

$$S_n^{i,k} := u^i \cdot \rho_n^k \cdot S_0, \quad (15)$$

where u^i captures the evolution due to the economic factors, and ρ_n^k captures the co-evolution with the money market interest rate.

3. Modeling the Stochastic Asset-Rate Covariance

In this section, we draw on the contango process ρ defined in Section 2 to build the multinomial hybrid lattice whose key property will be the stochastic asset-rate covariance.

3.1 Endogenous determination of the contango factor

In this subsection, we are interested in the determination of the contango factors $\{\Phi_n^{0,k}\}_{n,k}$. The following technical proposition provides an endogenous procedure to build this parametric structure via the principle of absence of arbitrage. As a result, the economic modeling of the asset-

rate covariance finds itself fully encapsulated in a single seeding value for each of the contango vectors $\{\Phi_n^{0,\cdot}\}_n$.

PROPOSITION 1 (Endogenous contango factor).

(a) *Under the absence of arbitrage opportunity, the following endogenous relationship holds:*

$$\Phi_n^{0,-1} = a_{n,0} + b_{n,0} \cdot \Phi_n^{0,1}, \quad (16)$$

where the coefficients $a_{n,0}$ and $b_{n,0}$ depend exclusively on r_0 , $r_{\pm 1}$, the underlying asset price diffusion parameters u and d , and the money market funding costs $R_{n,0}$ and $R_{n+1,0}$.

(b) *More generally, for all time steps n and all levels k of the short rate, the absence of arbitrage opportunity leads to the following endogenous relationship:*

$$\Phi_n^{k,k-1} = a_{n,k} + b_{n,k} \cdot \Phi_n^{k,k+1}, \quad (17)$$

where the coefficients $a_{n,k}$ and $b_{n,k}$ depend exclusively on the short rate at levels k , $k \pm 1$, the underlying asset price diffusion parameters u and d , and the term structure of the money market funding cost.

(c) *As a result, given the single value of the contango factor $\Phi_n^{0,1}$, the contango vector $\{\Phi_n^{0,k}\}_k$ becomes fully determined thanks to the endogenous Equation (17) and the recursive relationships (11), (12) and (13).*

Proof. See Appendix A. □

3.2 Geometry of the hybrid lattice

Our aim here is to determine the values of the transition multipliers $u_n^{0,k}$ (resp. $m_n^{0,k}$, $d_n^{0,k}$) from level r_0 , and then more generally the values of the transition multipliers $u_n^{j,k}$ (resp. $m_n^{j,k}$, $d_n^{j,k}$) from any level r_j . Doing so will enable us to determine both the underlying asset value and its evolution

lattice. The preceding subsection shows that the knowledge for all time-steps n of the seeding contango factors $\Phi_n^{0,1}$ renders the construction of all contango vectors $\{\Phi_n^{0,k}\}_k$ possible, as well as the determination of all transition multiplier vectors $\{u_n^{0,k}\}_k$ and $\{d_n^{0,k}\}_k$. Since Assumption 5 presupposes path convergence within the lattice, any node located within the lattice can be reached via different paths. As the following result shows, there is actually no need for further assumptions to determine all the hybrid transition multipliers $u_n^{j,k}$ and $d_n^{j,k}$.

PROPOSITION 2 (Transition multipliers). *The transition multipliers in the hybrid lattice are given for all states $(n; j, k)$ by:*

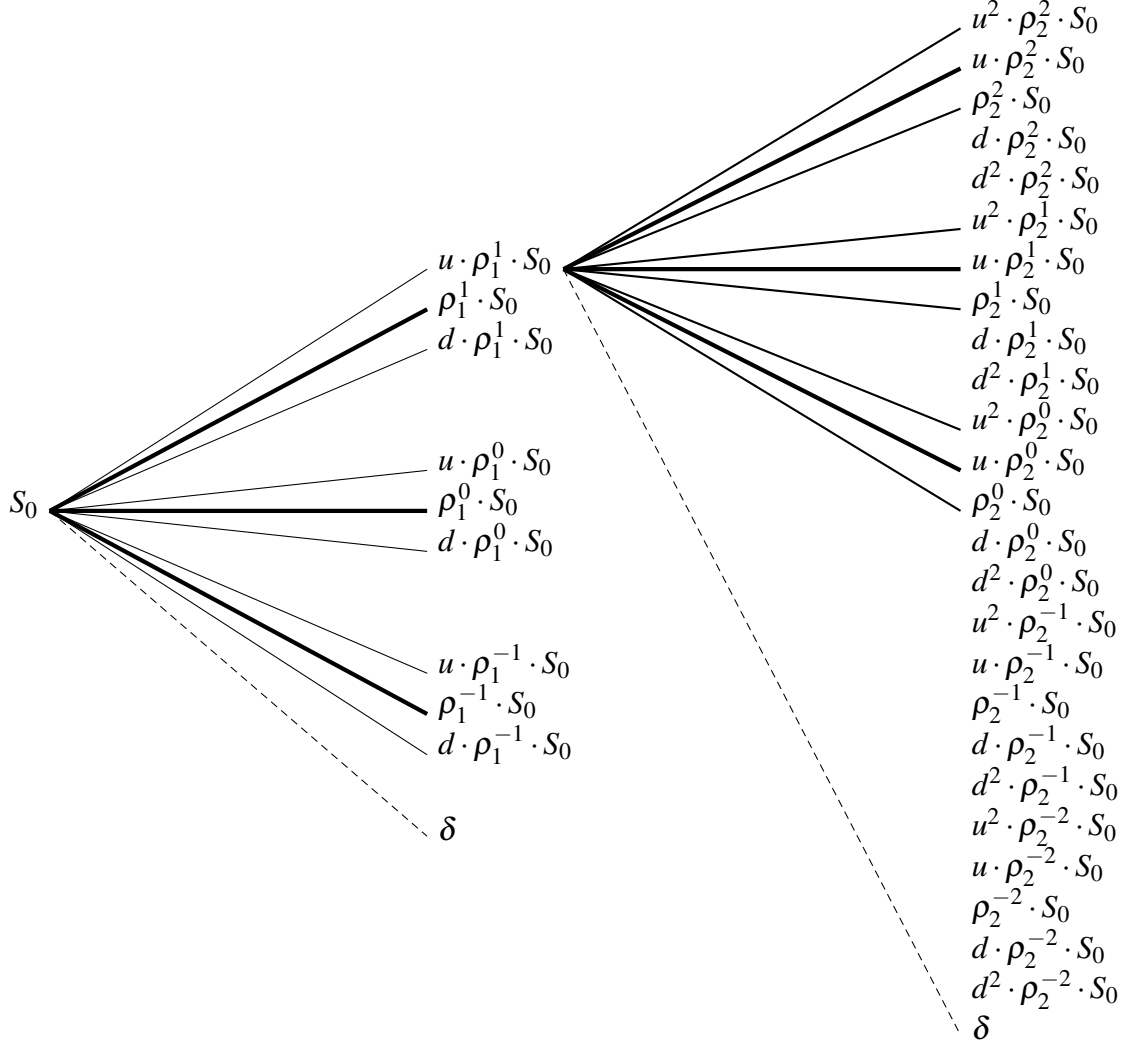
$$\begin{cases} u_n^{j,k} = u \cdot u_n^{0,k} / u_{n-1}^{0,j}, \\ m_n^{j,k} = \Phi_n^{0,k} / \Phi_{n-1}^{0,j}, \\ d_n^{j,k} = d \cdot d_n^{0,k} / d_{n-1}^{0,j}. \end{cases} \quad (18)$$

Proof. See Appendix B. □

Knowing the hybrid transition multipliers $u_n^{j,k}$ and $d_n^{j,k}$ enables us to deduce the hybrid lattice for the asset price. In summary, once the money market funding cost model is constructed, to graft the asset price quadrinomial lattice onto the interest rate trinomial lattice it is only necessary to know:

1. the value of the asset price bullish parameter u , in the case in which the interest rate stays constant at its initial level;
2. the various contango factors $\Phi_n^{0,1}$ for all time steps n , which, at each step n , indicate the underlying asset price sensitivity to the evolution of the money market interest rate.

Notice that the evolution of the contango process ρ_n^k captures all the information on the covariance between the joint evolution of asset prices and interest rates, and provides the hybrid lattice with its skeleton. The usual values of a classical trinomial model appear in the middle of the hybrid lattice (albeit weighted by the survival probability $1 - q_0$), in the case in which the money market funding cost remains constant at level r_0 . However, as soon as the money market funding cost changes, the asset price lattice is far from resembling the classical trinomial model.



3.3 Transition probabilities within the hybrid lattice

It is now possible to establish the transition probabilities from scenario $(n; i, j)$ to the ten subsequent scenarios that arise in the hybrid lattice at time-step $n + 1$. The asset price movement is assumed to result from two distinct effects —one concerning the economic factors, and the other concerning the money market interest rate. The probability of the evolution of the economic factors is assumed to depend on the level of the interest rate r at step n . Let $q_u^{n,j}$ (resp. $q_m^{n,j}$, $q_d^{n,j}$) be the probability of a bullish (resp. stable, bearish) evolution of the economic factors when going from step n to step $n + 1$ conditionally on the interest rate being r_j over this period. Generally speaking, the impact of the variations in the economic factors is not directly observable from the

asset price. Nevertheless, in the specific case of a constant short-term interest rate, the effect of the economic factors can be observed (Assumption 2). In such a case, the variation in the asset price is totally due to that of the economic factors, and the probability of an evolution in the asset price is exactly the same as the probability of an evolution in the economic factors. As a consequence, probabilities specific to each of the nine non-absorbing scenarios transitioning from r_j to r_k appear as the products of the three probabilities $q_u^{n,j}$, $q_m^{n,j}$ and $q_d^{n,j}$ with the three probabilities $p_n^{j,k-1}$, $p_n^{j,k}$ and $p_n^{j,k+1}$. Note that the probability of an evolution to the last absorbing scenario is q_0 which remains constant at each step. As a result, it is only necessary to specify the values of the marginal probabilities $q_u^{n,j}$, $q_m^{n,j}$ and $q_d^{n,j}$.

PROPOSITION 3 (Transition probabilities). *In the specific case where the interest rate stays at the level j between n and $n+1$ and the default state is absorbing ($\delta \equiv 0$), the marginal risk-neutral transition probabilities of the asset price lattice are given by:*

$$\begin{cases} q_u^{n,j} = \frac{V + M^2 + d(1 - q_0)m_j^2 - M(d + 1)m_j}{m_j^2(u - 1)(u - d)}, \\ q_d^{n,j} = \frac{V + M^2 + u(1 - q_0)m_j^2 - M(u + 1)m_j}{m_j^2(1 - d)(u - d)}, \\ q_m^{n,j} = 1 - q_0 - \frac{V + M^2 + (u + d - 1)(1 - q_0)m_j^2 - M(u + d)m_j}{m_j^2(1 - d)(u - 1)}, \end{cases} \quad (19)$$

where $M := (1 + r_j - q)\Delta t$ is the conditional mean of the asset price process in the money market account numeraire, $V := \sigma^2\Delta t$ is the conditional variance of the asset price process, and $m_j := m_n^{j,j}$ is the stable transition multiplier within the lattice at level j . In the non-hybrid and non-defaultable case (i.e., $j = 0$, $m_j = 1$ and $q_0 = 0$) we recover the standard trinomial transition probabilities of Boyle (1988).

Proof. See Appendix C. □

3.4 A hybrid diffusion with two processes and a stochastic covariance

The evolution of the asset price depends on two interdependent random processes, the economic factors that affect the underlying asset and the money market funding cost. As shown by the next result, the expression for the covariance between the asset price percentage change and the interest rate change can be obtained. It depends, notably, on the contango factors $\Phi_n^{0,k}$, that is, on the economic factors and interest rate processes. As a consequence, the covariance process is state-dependent and stochastic, which appears to be a sensible assumption when it comes to pricing futures contracts.

PROPOSITION 4 (Stochastic covariance). *The covariance process between the dynamics of the underlying asset's economic factors and the dynamics of the money market funding cost is stochastic.*

Proof. By definition, the asset-rate covariance is given by:

$$\text{cov} \left[\frac{\Delta S}{S}, \Delta r \right] = \mathbb{E} \left[\frac{\Delta S}{S} \cdot \Delta r \right] - \mathbb{E} \left[\frac{\Delta S}{S} \right] \cdot \mathbb{E} [\Delta r]. \quad (20)$$

The right-hand-side crossed expectation at node $(n; i, j)$ may be expressed using the definition (14) of the contango process ρ :

$$\sum_{k=j-1}^{j+1} p_n^{j,k} \frac{q_u^{n,j} S_0 u^i (u \rho_n^k - \rho_n^j) + q_m^{n,j} S_0 u^i (m \rho_n^k - \rho_n^j) + q_d^{n,j} S_0 u^i (d \rho_n^k - \rho_n^j)}{S_0 u^i \rho_n^j} (r_k - r_j). \quad (21)$$

Breaking down the double source of risk as follows:

$$u \rho_n^k - \rho_n^j = u \rho_n^k - u \rho_n^j + u \rho_n^j - \rho_n^j = u(\rho_n^k - \rho_n^j) + \rho_n^j(u - 1), \quad (22)$$

this crossed expectation becomes:

$$\sum_{k=j-1}^{j+1} p_n^{j,k} \frac{q_u^{n,j} u \Delta \rho_n^k + q_m^{n,j} m \Delta \rho_n^k + q_d^{n,j} d \Delta \rho_n^k}{\rho_n^j} (r_k - r_j) + \sum_{k=j-1}^{j+1} p_n^{j,k} (q_u^{n,j} \Delta u + q_d^{n,j} \Delta d) (r_k - r_j). \quad (23)$$

where $\Delta \rho_n^k := \rho_n^k - \rho_n^j$, $\Delta u := u - 1$ and $\Delta d := d - 1$. We observe now that the second sum is the product of expectations $E[\Delta S/S] \cdot E[\Delta r]$ at node $(n; i, j)$. Substituting into the covariance definition (20), we obtain at node $(n; i, j)$:

$$\text{cov} \left[\frac{\Delta S}{S}, \Delta r \right] = \sum_{k=j-1}^{j+1} p_n^{j,k} \frac{q_u^{n,j} u + q_m^{n,j} m + q_d^{n,j} d}{\rho_n^j} \Delta \rho_n^k (r_k - r_j) \quad (24)$$

$$= \frac{q_u^{n,j} u + q_m^{n,j} m + q_d^{n,j} d}{\rho_n^j} \Delta r^2 \cdot E \left[\frac{\Delta \rho}{\Delta r} \right]. \quad (25)$$

The asset-rate covariance is thus stochastic since it fluctuates according to the position in the hybrid lattice. \square

We now turn back to the designation of $\Phi_n^{0,1}$ as a contango factor. The following result shows that this terminology is consistent with that commonly used in futures markets.

PROPOSITION 5 (Contango regime). *For a continuous-time, continuous-state economy and under the assumption that the underlying asset makes no discrete payouts, the futures price is an increasing function of the asset-rate covariance. Moreover, as soon as the contango process is an increasing function of the short rate (i.e., $\Delta \rho / \Delta r \geq 0$), the futures price will exceed the expected future spot price, thereby leading to a contango regime.*

Proof. We know from Cox, Ingersoll, and Ross (1981) that under the condition that the underlying asset pays no discrete payouts, the futures price is a decreasing function of the local covariance between the percentage change in the underlying asset price with the percentage change in the risk-free discount bond price. More precisely, if $F(t, T)$ denotes the t -time futures price for maturity T and $P(u)$ is the price at time u of a default-free discount bond paying one dollar at time T ,

Equations (17) and (25) of Cox, Ingersoll, and Ross (1981, Proposition 9 p.329) show that :

$$F(t, T) = E[S_T] + e^{\int_t^T \ln(1+r_u) du} \left(\int_t^T \frac{S_u}{P(u)} \cdot \text{var} \left[\frac{\Delta P}{P} \right] du - \int_t^T \frac{S_u}{P(u)} \cdot \text{cov} \left[\frac{\Delta S}{S}, \frac{\Delta P}{P} \right] du \right) \quad (26)$$

Noticing that $\Delta P/P = -\Delta r$, we can substitute the local covariance between the percentage change in the underlying asset with the change in the interest rate, which yields:

$$F(t, T) = E[S_T] + e^{\int_t^T \ln(1+r_u) du} \left(\int_t^T \frac{S_u}{P(u)} v^2 du + \int_t^T \frac{S_u}{P(u)} \cdot \text{cov} \left[\frac{\Delta S}{S}, \Delta r \right] du \right). \quad (27)$$

Every quantity appearing on the right-hand-side is always positive, except $\text{cov}[\Delta S/S, \Delta r]$. As a result, the futures price $F(t, T)$ becomes an increasing function of the asset-rate covariance.

The second part of Proposition 5 comes in a straightforward way from Equation (25) which ensures that the sign of the local asset-rate covariance is determined by the sign of the local rate of variation $\Delta \rho / \Delta r$. □

3.5 Pricing futures contracts

The futures price being merely defined as the delivery price for which the value of the futures contract is zero, it is not the value of a financial asset in itself. As such, futures prices cannot be directly valued in our arbitrage-free framework. However, as recognized by Cox, Ingersoll, and Ross (1981, Proposition 7), the inception price of a futures contract is also the value of a specific financial asset that would pay the underlying asset price at maturity, as well as a continuous flow of the prevailing spot rate times the prevailing futures price from inception up to the futures' maturity. The latter contract is no different from the underlying asset paying an extra continuous dividend at the money-market, risk-free rate. It turns out that this new asset is easily valued in our framework, whose numeraire is the money market account. As a result, the desired futures price will be obtained by regular backward induction of the underlying asset terminal price, as soon as we cease to discount cash-flows at each time-step of the hybrid lattice.

4. Application to the Valuation of Financial Futures

In this section, we use our discrete-time valuation framework to carry out numerical simulations for the purpose of conducting a sensitivity analysis of futures contracts.

4.1 Pricing assumptions

We use the methodology described in Section 3.5 to price futures contracts. More precisely, backward induction without stepwise discounting is applied to the terminal distribution of the asset price along the hybrid lattice. At this stage, it is worth emphasizing the non-explosive aspect of the model that arises from the parsimonious properties of the Hull and White short rate tree, which unfolds like a “tube” rather than a cone. At time-step n , the total number of scenarios is $(2n + 1)j_n$, where j_n is the number of interest rate scenarios that are specific to the time-step. Hull and White (1994) show that the maximal interest rate index may be chosen as $j_{max} = \lfloor 0.184/(a\Delta t) \rfloor$, a choice that ensures a bounded number of interest rate scenarios. As the size of the hybrid lattice grows only linearly with the time variable, we are able to use a time resolution as fine as a day for shorter-term maturities (such as 3-month contracts) and no longer than a week for longer-term maturities such as 5-year contracts (with a simulation covering 260 weekly periods). We optimize the spatial resolution with a stretch parameter of $\theta = \sqrt{3}$ as described in Boyle (1988) and Hull and White (1990a). The spatial step Δr used for the interest rate is then equal to $v\sqrt{3\Delta t}$, and the underlying asset transition multiplier u is equal to $\exp(\sigma\sqrt{3\Delta t})$.

Table 1 lists the pricing assumptions and market data used in our simulations. We used a flat money market interest rate curve set at 2% per annum. Without loss of generality, we assumed no discrete cash dividend payment paid out by the underlying asset.

Table 1. Pricing assumptions used in numerical simulations

Parameter	Description	Value
S_0	Underlying asset initial price	\$100.00
σ	Underlying asset volatility (annualized)	20.0%
q	Underlying asset dividend yield & repo rate	0%
λ	Underlying asset default intensity	0%
r	Risk-free interest rate	2.00% per annum
a	Interest rate mean reversion speed	0.1
v	Interest rate normal volatility (annualized)	0.1%

4.2 The impact of interest rate variations on the underlying asset price

The role of the contango factor is to amplify or diminish the impact of the changes in the interest rate on the simulated asset price. When $\Phi_n^{0,k}$ is equal to one, the impact is strictly proportional to the ratio $(1 + r_0)/(1 + r_k)$. When $\Phi_n^{0,1}$ is greater than one, the high comovement of asset prices and interest rates dampens the impact of an interest rate increase, and highly-covaryed asset values lead to a reduced asset price dispersion across the hybrid lattice. By contrast, when $\Phi_n^{0,1}$ is less than one, the low comovement of asset prices and interest rates amplifies the impact of an interest rate increase, and the asset price dispersion turns out to be accentuated.

Figure 1a illustrates the lattice for the contango factor as calculated by the model, under the assumption $\Phi_n^{0,1} \equiv 1.0015$. In the region of high interest rate levels ($k \uparrow j_{max}$), high contango factors are endogenously generated to counterbalance the distortive effects of high interest rates on the asset price. Conversely, low interest rate levels ($k \downarrow j_{min}$) give rise to endogenous contango factors lower than one which act as counterweights to possibly negative interest rates. Figure 1b plots the cross-section of the contango process, ρ_{60}^k . The slight deviation from the contango factor cross-section, $\Phi_{60}^{0,k}$, is due to the factor $(1 + r_0)/(1 + r_k)$. We observe that the seeding choice of $\Phi_n^{0,1} \gg 1$ leads to a monotonically increasing contango process that is synonymous with a contango regime (see Proposition 5).

Figure 1. The contango factor lattice

This figure plots the contango factor lattice as calculated by the model with 60 monthly time-steps in the simulation. The seeding value $\Phi_n^{0,1}$ has been set to 1.0050 for all time-steps n ($1 \leq n \leq 60$). The parameter $\Phi_n^{0,j_{max}}$ (resp. $\Phi_n^{0,j_{min}}$) appears on the higher (resp. lower) edge of the surface at each time-step. The interest rate volatility ν has been set to 1%, leading to $j_{max} = |j_{min}| = 23$. Other parameters are set out in Section 4.1.

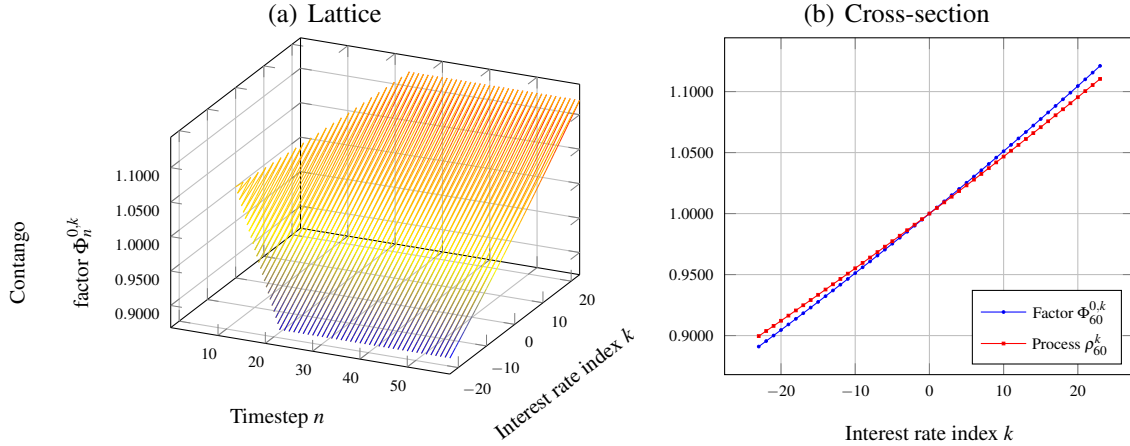
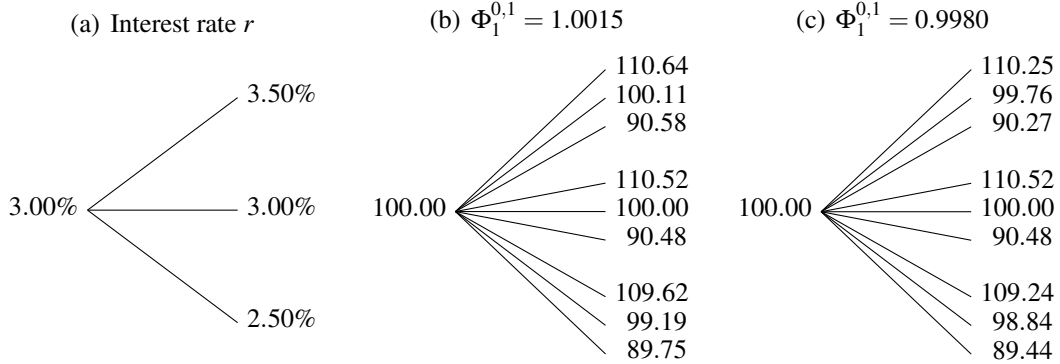


Figure 2 shows the interest rate and the asset price values at the first time-step of our hybrid lattice under the assumption of a dampening contango factor $\Phi_1^{0,1} = 1.0015$. The value 0.9915 of the endogenous parameter $\Phi_1^{0,-1}$ is calculated by the model (see Proposition 1). As with parameter $\Phi_1^{0,1}$, this dampening value reduces the impact of the interest rate variation on the asset price. Note that lower values of the parameter $\Phi_1^{0,1}$ coupled with an increase in the interest rate would lead to lower values of the underlying asset price. Lastly, right in the middle of the asset price tree, the value of the parameter $\Phi_1^{0,0}$ is equal to unity and the asset price values are exactly the same as those in the standard trinomial model (Boyle (1988)). Figure 2 also shows the asset price values at the first time-step of our hybrid lattice for an amplifying contango factor $\Phi_1^{0,1} = 0.9980$. If interest rates are rising, then choosing contango factors inferior to one provide results that better conform to expectations by amplifying the downward movement caused by this trend in interest rates.

Figure 2. The underlying asset price lattice at the first time-step

This figure shows the values at the nine nodes of the first monthly time-step of our multinomial lattice. (a) The interest rate r_k , (b) the asset price $S_1^{i,k}$ when $\Phi_1^{0,1} = 1.0015$, and (c) the asset price $S_1^{i,k}$ when $\Phi_1^{0,1} = 0.9980$. The underlying asset's economic factors index takes three possible values ($i = -1, 0, 1$). The interest rate index takes three possible values ($k = -1, 0, 1$). The interest rate volatility ν has been set to 1%. Other parameters are set out in Section 4.1.



4.3 The impact of the contango factor on the spot-futures basis

We now apply our two-factor discrete-time model to the valuation of financial futures contracts. The theoretical futures price is calculated as the expected asset terminal value given by our hybrid lattice without stepwise discounting (see Section 3.5). Comparisons will be made against the corresponding unique forward price obtained from the traditional cost-of-carry model, which can be considered an unbiased estimator of the expected future spot price. As a result, the deviation of the theoretical futures price from this unique forward price may be viewed as a proxy for the spot-futures basis.

Table 2 reports theoretical futures prices for short-, medium- and long-term futures maturities. The deviations from the unique forward price at each maturity are reported as percentages. As awaited, the futures price appears to be (i) an increasing function of the contango factor, (ii) greater than the expected future spot price (contango) when $\Phi^{0,1} > 1$, (iii) less than the expected future spot price (normal backwardation) when $\Phi^{0,1} < 1$. For contango factor values close to unity, deviations from forward prices remain less than 1%. As soon as the contango factor moves away from one, however, we observe significant deviations larger than 1% from the traditional cost-of-carry model

of futures. More importantly, we can observe theoretical futures prices below the underlying spot price for short-term maturities of less than 6 months, even though a substantial risk-free interest rate (3% per annum) is used for these numerical simulations.

Table 2. Impact of the contango factor on the theoretical futures price

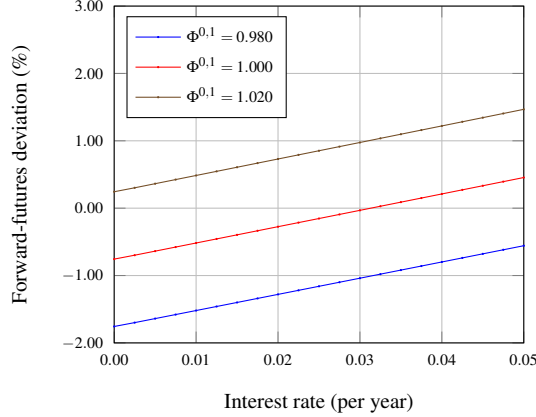
This table reports the model theoretical futures prices and its deviation in percent from the corresponding unique forward price (obtained from the cost-of-carry model) for short-, medium- and long-term contract maturities. Futures prices are functions of the contango factor $\Phi^{0,1}$ (assumed constant at each time-step of the simulation). The underlying asset's initial price is $S_0 = \$100.00$ and the interest rate term structure is flat at 3% per annum. Other pricing assumptions are set out in Section 4.1.

$\Phi^{0,1}$	Maturity of futures							
	3 months		6 months		1 year		5 years	
	Price (\$)	Dev. (%)	Price (\$)	Dev. (%)	Price (\$)	Dev. (%)	Price (\$)	Dev. (%)
0.980	99.74	(1.01)	100.44	(1.02)	102.00	(1.02)	114.95	(1.07)
0.985	99.99	(0.76)	100.69	(0.77)	102.26	(0.77)	115.24	(0.82)
0.990	100.24	(0.51)	100.95	(0.52)	102.51	(0.52)	115.53	(0.57)
0.995	100.49	(0.26)	101.20	(0.27)	102.77	(0.27)	115.82	(0.32)
1.000	100.74	(0.01)	101.46	(0.02)	103.03	(0.02)	116.12	(0.07)
1.005	100.99	0.24	101.71	0.23	103.29	0.23	116.41	0.18
1.010	101.25	0.49	101.96	0.48	103.54	0.48	116.70	0.43
1.015	101.50	0.74	102.22	0.73	103.80	0.73	116.99	0.68
1.020	101.75	0.99	102.47	0.98	104.06	0.98	117.28	0.93

Figure 3 plots the deviation of the theoretical 3-month futures price from the unique forward price as a function of the interest rate and for various levels of the contango factor. As predicted by Cox, Ingersoll, and Ross's (1981) theory and Equation (27), the forward-futures price deviation is an increasing function of the interest rate. However, the level of the implicit asset-rate covariance built in the contango factor appears to have a steep impact on the level of the forward-futures deviation. The higher the contango factor, the more beneficial the effects on the margin account of the holder of a long futures position, and the higher the forward-futures deviation. Figure 3 confirms that a wide spectrum of scenarios for the spot-futures basis may thus be covered by our model owing to the flexibility of the Φ parameter.

Figure 3. Impact of the contango factor on the forward-futures deviation

This figure plots the deviation of the 3-month futures price from the corresponding unique forward price (obtained from the cost-of-carry model) as a function of the interest rate assumed constant at each time-step of the simulation. The interest rate volatility has been set to $v = 0.1\%$. Other pricing assumptions are set out in Section 4.1.



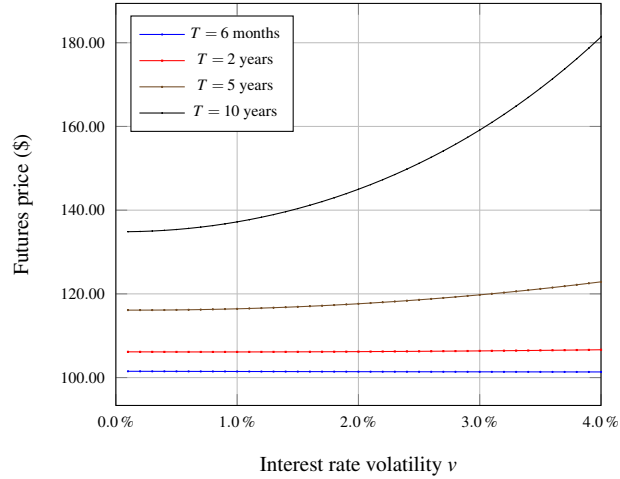
4.4 The impact of the interest rate volatility on futures prices

In a context of stochastic interest rates, we know from Cox, Ingersoll, and Ross (1981) and Equation (27) that the futures price is an increasing function of the variance of interest rates. Put differently, a higher interest rate volatility should translate into a higher comovement between interest rates and asset prices, thereby pushing the futures price upward.

Figure 4 plots the futures price given by our two-factor model as a function of the interest rate volatility v . A neutral contango factor ($\Phi^{0,1} = 1$) has been chosen to run the simulations. As the CIR theory predicts, the futures price appears to be an increasing function of interest rate volatility, an effect which becomes more significant for longer-term maturities.

Figure 4. Impact of the interest rate volatility on the theoretical futures price

This figure plots the model theoretical futures price as a function of the interest rate volatility assuming a neutral contango factor $\Phi^{0,1} = 1.00$. The futures contract time to maturity T has been varied from short-term maturities (6 months) to longer-term maturities (10 years). The underlying asset's initial price is $S_0 = \$100.00$ and the interest rate term structure is flat at 3%. Other pricing assumptions are set out in Section 4.1.



4.5 The impact of the contango factor on futures options

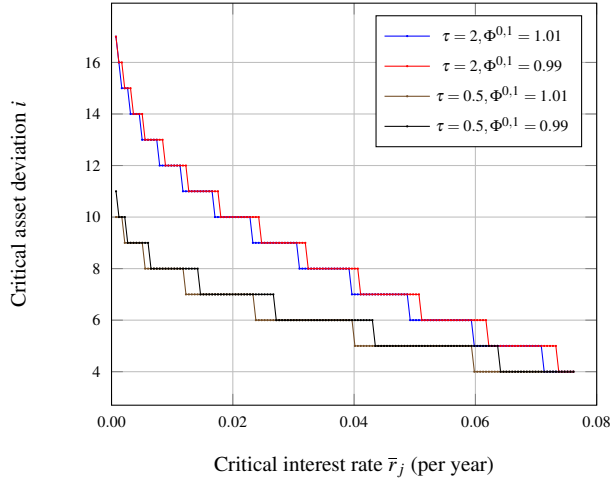
The buyer of a call (resp. put) option on a futures contract is entitled to receive (resp. pay) the difference in cash between the most recently settled futures price and the option's exercise price, and simultaneously to enter into a long (resp. short) position in the futures contract at the most recent settlement price. It is well known that, contrary to American-style call options on an underlying asset paying no dividend, American-style call options on futures contracts are prone to premature exercise (see, for example, Ramaswamy and Sundaresan, 1985). The reason for this counterintuitive fact lies in the “implicit dividend” continuously paid by a futures contract at the short-term interest rate (see Section 3.5). Indeed, as soon as the present value of future implicit dividends exceeds the interest that can be earned on the exercise price, the force of interest renders optimal the early exercise of a call option written on the futures contract. As a result, the dynamics of interest rates play a major role when it comes to characterizing the optimal exercise policy of futures options.

Figure 5 shows the two-dimensional critical region $\{\bar{r}, \bar{S}\}$ for American-style call options on

a stock index futures, as predicted by the model for two different time horizons (6 months and 2 years). For the sake of clarity, the option and the futures contract have the same time to maturity, τ , and the early exercise boundary $\bar{S}_n^{i,j} = S_0 \rho_n^j \exp\{i\sigma\sqrt{\Delta t}\}$ has been expressed in the number i of possibly negative standard deviations from the option's exercise price set equal to S_0 . We observe that the present value of the futures contract's implicit dividends increases with the critical short-term interest rate \bar{r}_j , thereby lowering the critical futures price. As a result, the critical asset price $\bar{S}_n^{i,j}$ at which early exercise becomes optimal is lowered, as well. The early exercise boundary $\bar{S}(\bar{r})$ is therefore a decreasing function of the critical short-term interest rate \bar{r} for each maturity, a general result in line with previous studies (e.g., Ramaswamy and Sundaresan, 1985). Our findings suggest, however, that the role played by the contango factor is just as determinant. When the latter is greater than one, highly-covariating interest rates and asset values produce higher futures prices. Early exercise then becomes optimal at a lower critical futures price, giving rise to a lower critical pair $\{\bar{r}, \bar{S}\}$ in return. Consequently, the early exercise boundary measured in the asset price metrics appears to be significantly lowered. Conversely, when the contango factor is less than one, the model produces smaller futures prices, which give rise to higher critical futures prices and a higher early exercise boundary.

Figure 5. Optimal exercise boundary for call options on futures contracts

This figure plots the optimal exercise boundary $\{\bar{r}, \bar{S}\}$ for American-style call options on a futures contract of same maturity τ as a function of the critical short-term interest rate \bar{r} . The critical asset price $\bar{S}_n^{i,j}$ is measured in the number i of (possibly negative) standard deviations from the option's exercise price set to $K = \$100.00$. The interest rate volatility has been set to $\nu = 0.2\%$. Other pricing assumptions are set out in Section 4.1.



5. Empirical Analysis

In this section, we apply our theoretical model of futures pricing to empirical data.

5.1 The data

We consider two major stock market indexes (S&P 500, Euro Stoxx 50) which underlie the most heavily traded stock index futures contracts in the world. Futures contracts on these two indexes present interesting dissimilarities concerning both their costs of marking to market and their costs of carry. S&P 500 futures have recently evolved in a context of (i) rising short-term interest rates on the money market (caused by the US Federal Reserve's recent monetary policy tightening), and (ii) uniform costs of carry across futures expiries (the S&P 500 index enjoys a broad and diversified base of quarterly-dividend-paying stocks). By contrast, Euro Stoxx 50 futures still evolve in a context of negative short-term interest rates. Moreover, dividend uncertainty on

the Euro Stoxx 50 is concentrated between March and June futures expiries.⁵ This fact leads to a low and highly predictable cost of carry during the rest of the year.

Spot and futures prices are expressed in index points. We consider daily observations from December 2015 to December 2017. Although a futures contract may be opened up to a year before its delivery month,⁶ market participants are predominantly invested into the first nearby (or “front month”) contract, and then roll into the next nearby contract over the few days preceding its expiration date. As a consequence, we limit our observations to the 4-month period of active trading before the rollover date, the only time period during which significant open interest⁷ is observed on the futures contract. Data are from Thomson Reuters.

Table 4 reports summary statistics for the daily spot-futures basis, $S(t) - F(t, T)$, and the daily futures risk premium, $F(T, T) - F(t, T)$, as well as the correlation between both variables. In the case of the S&P 500, we observe a slightly decreasing trend in the mean of the spot-futures basis that will be interpreted later in relation to the monetary policy (see Section 5.4). Notice how the basis standard deviation is low and does not differ much across contracts. In contrast, the variability of the futures risk premium is much more pronounced. We notice also that the empirical correlation between the spot-futures basis and the futures risk premium never exceeds 76%. This level appears to be lower than what might be expected between these two structurally connected variables. Finally, the Euro Stoxx 50 spot-futures basis and futures risk premium appear significantly more variable than their S&P 500 counterpart.

⁵Approximately 80% of the Euro Stoxx 50 index’s dividend volume is paid between March and June as most European blue-chip companies pay their single annual dividend in the spring.

⁶Expirations for stock index futures contracts usually occur on the third Friday of the delivery months that follow a quarterly expiration cycle (i.e., March, June, September and December).

⁷The open interest of a futures contract is the number of outstanding contracts that have not been closed by an offsetting trade.

Table 3. Descriptive statistics

This table reports summary statistics for the daily spot-futures basis, $S(t) - F(t, T)$, and the daily futures premium, $F(T, T) - F(t, T)$, as well as their correlation. Time period: December 2015 to December 2017. Data source: Thomson Reuters.

Contract	S&P 500					Euro Stoxx 50					Obs.
	Basis (pts)		Premium (pts)		Corr.	Basis (pts)		Premium (pts)		Corr.	
	Mean	SD	Mean	SD		Mean	SD	Mean	SD		
Mar. 16	6.7	4.4	73.0	77.3	-0.09	4.3	22.3	-38.5	210.3	-0.16	82
Jun. 16	6.4	4.1	37.1	53.4	0.76	52.5	30.7	-89.1	80.4	0.27	82
Sep. 16	6.0	4.6	15.2	51.6	0.70	5.8	13.8	10.1	88.5	0.05	82
Dec. 16	5.4	3.4	106.6	37.3	0.49	7.7	12.9	240.2	59.5	0.44	82
Mar. 17	4.2	2.6	108.1	61.6	0.72	5.4	8.9	203.2	112.0	0.48	82
Jun. 17	3.3	2.4	57.8	28.8	0.50	52.0	29.1	98.1	123.4	0.83	82
Sep. 17	2.0	2.0	54.2	27.7	0.30	7.1	10.2	34.0	50.2	-0.37	82
Dec. 17	1.9	2.1	124.6	63.1	0.20	6.8	8.1	-3.0	77.6	0.46	82

5.2 Testing for the model robustness

In the sequel, we test the robustness of our discrete-time model by calibrating the contango factor against the closing price histories of the futures contract and its underlying cash index. We obtain the market-implied contango factor $\tilde{\Phi}$ as the value of the parameter $\Phi_1^{0,1}$ such that the model yields the closing futures price for the given simultaneous spot price of the cash index. Being an explicit function of simultaneous closing spot and futures prices, the market-implied contango factor is thus an implicit function of the spot-futures basis:

$$\tilde{\Phi}(t, T) = F_{\sigma, \nu, a}(S(t) - F(t, T), r_t, d_{t, T}), \quad (28)$$

where σ, ν and a are the diffusive parameters of the model, r_t is the interest rate term structure, and $d_{t, T}$ is the dividend forecast for horizon T seen from date t . Notice how the quantity $\tilde{\Phi}(t, T)$ subsumes the daily level of the spot-futures basis altogether with the market's interest rate expectations. Akin to the implied volatility of option contracts, the market-implied contango factor provides market participants with a universal gauge of the level of contango that is consistent across futures markets and maturities.

The calibration of the market-implied contango factor entails some assumptions regarding the estimation of the cost of carry of stock index futures. In fact, since the 3-month period between rollover dates is the only time period during which significant open interest is observed, the cost of carrying the front-month contract may be assumed to be deterministic in a first approximation. Indeed, as dividend uncertainty quickly resorbs when moving closer to the delivery date, the dividend forecast $d_{t,T}$ expected by futures arbitrageurs adjusts with the dividend yield that is effectively realized by the stock market index. As a result, we estimate the cost of carrying front-month futures by means of the index of dividend points effectively realized by the underlying cash index. Moreover, to ensure a consistent calibration of $\tilde{\Phi}$ across dates, all calibrations are performed with the same constant volatility parameter σ for the diffusion of the underlying cash index, as well the same constant diffusive parameters ν and a for the interest rate diffusion.

Figure 6 displays the market-implied contango factor $\tilde{\Phi}$ for both the S&P 500 and the Euro Stoxx 50 futures contracts expiring in March 2017. On each trading day, $\tilde{\Phi}$ has been calibrated against the futures and cash index closing prices. In a sign of model robustness, the calibration shows very little sensitivity to the levels of the cash index volatility and the interest rate volatility, which led us to calibrate with standard values of these diffusion parameters ($\sigma = 14\%$, $\nu = 0.1\%$, $a = 0.1$).

Figure 6. Contango factor implied from stock index futures

This figure plots as a function of time the daily contango factor $\tilde{\Phi}$ (bullet) and the daily spot-futures basis (triangle) implied from: (a) the E-Mini S&P 500 futures maturing in March 2017, (b) the Euro Stoxx 50 futures maturing in March 2017. On each trading day, $\tilde{\Phi}$ has been calibrated against the closing prices of the futures contract and the cash index. Futures' costs of carry have been estimated from the cumulated indexes of annual realized dividend points. Overnight, 1-week, 1-month, 2-month and 3-month tenors of LIBOR (resp. EURIBOR) interest rates have provided proxies for the USD (resp. EUR) money market. Constant volatility parameters have been used to diffuse the underlying cash index price process ($\sigma = 14\%$) and the interest rate process ($\nu = 0.1\%$, $a = 0.1$). Data source: Thomson Reuters.

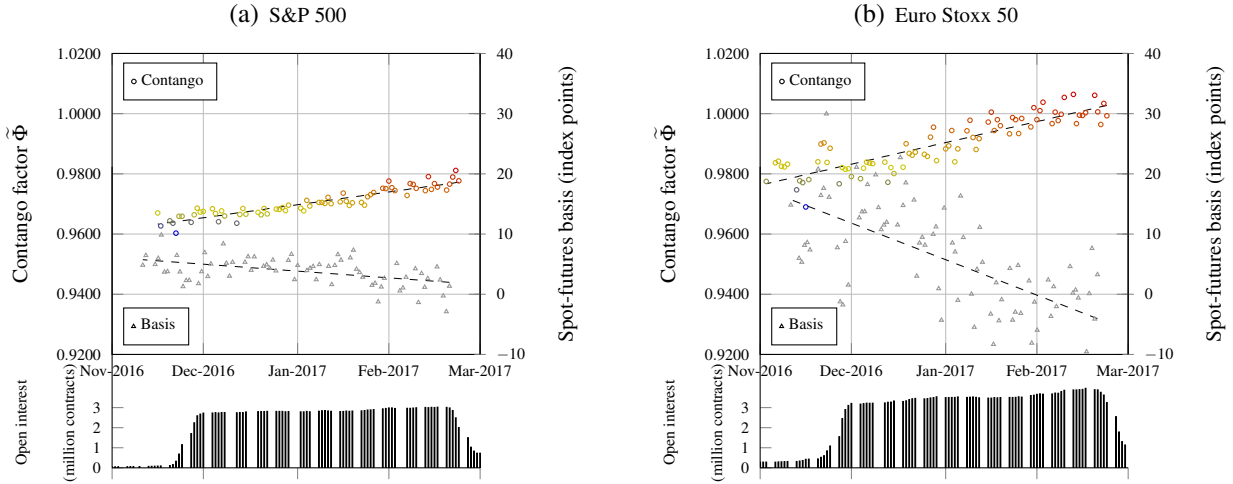


Figure 6 also plots the spot-futures basis, in which the observed regime of normal backwardation may be explained by futures' high cost of carry. The implied contango factor increases slowly toward one, in accordance with our theoretical simulations which predict a contango less than one in case of normal backwardation of futures prices (see Section 4.3). But a second effect may be advanced to explain the convergence of $\tilde{\Phi}(t, T)$ toward one as the futures delivery approaches ($t \uparrow T$). Because the residual time to maturity of the futures contract shrinks to zero, the expected interest charges credited on the futures holder's margin account have less and less duration. As a result, the negative covariance between the underlying cash index and the overnight interest rate exerts a receding impact upon the formation of the futures price.

Table 4 reports summary statistics for the daily market-implied contango factor, $\tilde{\Phi}(t, T)$. Generally speaking, the market-implied contango factor appears to be stable along time, with low standard deviations. During the 2-year time period under scrutiny (2016-2017), the macroeconomic

context of low interest rates entailed a regime of normal backwardation on stock index futures, thereby forcing contango factors to evolve below unity. The third column of the S&P 500 panel displays the same reversed trend in the mean as the one observed in Table 4 for the spot-futures basis, in relation to the tightening of the monetary policy. The penultimate column highlights this negative correlation with the spot-futures basis. More importantly, this correlation is significantly different from -1, thereby highlighting the role of interest rate expectations in the calibration process of $\tilde{\Phi}(t, T)$. Finally, the descriptive statistics do not detect the existence of skewness (positive in six cases out of eight) or excess kurtosis. The Jarque-Bera test confirms that there are no significant departures from the hypothesis of a Gaussian distribution, warding off the prospect of heteroskedasticity in the data.

Table 4. Summary statistics of the daily market-implied contango factor

This table reports summary statistics for the daily market-implied contango factor during the period December 2015 to December 2017. On each trading day, $\tilde{\Phi}$ has been calibrated against the closing prices of the futures contract and the cash index. The costs of carrying futures have been estimated from the cumulated indexes of annual realized dividend points. Overnight, 1-week, 1-month, 2-month and 3-month tenors of LIBOR (resp. EURIBOR) interest rates have provided proxies for the USD (resp. EUR) money market. Constant volatility parameters have been used to diffuse the underlying cash index price process ($\sigma = 14\%$) and the interest rate process ($v = 0.1\%$, $a = 0.1$). *JB* is the Jarque-Bera statistic testing for normality. ***, ** and * denote statistical significance at the 1%, 5%, and 10% levels, respectively.

Contract	Min	Mean	Max	SD	Skewness	Kurtosis	<i>JB</i> stat.	Corr. w/ basis	Obs.
<i>Panel A: S&P 500</i>									
Mar. 16	0.936	0.955	0.977	0.011	0.38	2.03	5.18*	-0.80	82
Jun. 16	0.938	0.960	0.974	0.008	-0.29	2.28	2.93	-0.95	82
Sep. 16	0.945	0.958	0.971	0.007	-0.28	1.90	5.18*	-0.92	82
Dec. 16	0.942	0.955	0.968	0.007	0.08	1.86	4.51	-0.92	82
Mar. 17	0.942	0.953	0.966	0.006	0.05	2.10	2.82	-0.83	82
Jun. 17	0.954	0.963	0.974	0.005	0.49	2.16	5.69*	-0.82	82
Sep. 17	0.963	0.972	0.983	0.006	0.26	2.10	3.65	-0.69	82
Dec. 17	0.971	0.981	0.992	0.005	0.34	3.27	1.10	-0.54	82
<i>Panel B: Euro Stoxx 50</i>									
Mar. 16	0.942	0.985	1.027	0.018	-0.22	2.43	1.80	-0.91	82
Jun. 16	0.924	0.953	1.020	0.017	0.99	4.37	19.73***	-0.95	82
Sep. 16	0.950	0.985	1.025	0.013	0.05	2.98	0.03	-0.89	82
Dec. 16	0.953	0.984	1.011	0.013	-0.17	2.45	1.45	-0.90	82
Mar. 17	0.962	0.986	1.006	0.011	0.00	1.84	4.58	-0.89	82
Jun. 17	0.946	0.968	1.008	0.018	0.72	2.04	10.18***	-1.00	82
Sep. 17	0.976	0.992	1.009	0.008	-0.02	2.41	1.18	-0.93	82
Dec. 17	0.979	0.995	1.011	0.006	0.24	3.78	2.81	-0.98	82

5.3 Testing the explanatory power of the contango factor on the futures risk premium

We are now in a position to test the role of the daily market-implied contango factor, $\tilde{\Phi}(t, T)$, in the formation of the daily futures risk premium, which is understood as the difference between the futures price, $F(t, T)$, and a proxy at time t of the expected future spot price at the maturity of the futures contract, T . In order to isolate the role played by $\tilde{\Phi}$, we need to soak up any variability in the futures risk premium arising from market liquidity effects. This constraint suggests controlling for the depth and the trading activity of the futures market. We thus consider the following multivariate linear regression model:

$$F(T, T) - F(t, T) = \beta_0 + \beta_1 \cdot \tilde{\Phi}(t, T) + \beta_2 \cdot \text{OpInt}(t) + \beta_3 \cdot \text{Vol}(t) + \varepsilon_t, \quad (29)$$

where $\text{OpInt}(t)$ is the open interest at date t on the futures contract maturing in T , and $\text{Vol}(t)$ is the total buying plus selling volume at date t on the futures contract maturing in T . In specification (29), the terminal value of the futures price at the maturity of the contract, $F(T, T)$, provides the proxy for the future spot price expected on date t . The null hypothesis is $H_0 : \hat{\beta}_1 = 0$, that is, the contango factor implied from the market at time t contains no information about the concomitant futures risk premium. This hypothesis is tested against the alternative that the contango factor retains some explanatory power.

Table 5 summarizes the qualitative results from the ordinary least squares (OLS) regression (29) carried out on eight S&P 500 (panel A) and Euro Stoxx 50 (panel B) futures contracts. The regression sample covers the 4-month time period over which significant open interest in the stock index futures contract is observed (82 observations). Notice that the spot-futures basis is prone to autocorrelation by construction, which entails serially correlated residuals. As a result, robust standard errors corrected for heteroskedasticity and serial correlation by the Newey-West method are reported in Table 5. Due to the macroeconomic context of low interest rates during the 2-year time period under scrutiny (2016-2017), we expect negative slope coefficients for specification (29) as a confirmation that the futures risk premium dwindles over time, while the contango factor

gradually reverts to unity at the same time. Indeed, panels A and B show that negative slope coefficients ($\hat{\beta}_1$) for specification (29) are statistically significant at the 1% level for a majority of S&P 500 and for half of the Euro Stoxx 50 contracts. With most coefficients of determination (R_1^2) above 0.60, there is suggestive evidence that, for these futures contracts, the contango factor plays some role in the formation of the daily futures risk premium.

Table 5. OLS regressions of the futures risk premium against the contango factor

This table reports OLS estimates of the slope coefficients in the linear regression model:

$$F(T, T) - F(t, T) = \beta_0 + \beta_1 \cdot \tilde{\Phi}(t, T) + \beta_2 \cdot \text{OpInt}(t) + \beta_3 \cdot \text{Vol}(t) + \varepsilon_t, \quad (29)$$

t -statistics (not reported) are calculated using robust standard errors (reported in parentheses) corrected for heteroscedasticity and autocorrelation using the Newey-West method. ***, ** and * denote statistical significance at the 1%, 5%, and 10% levels, respectively. Data source: Thomson Reuters.

Contract	$\hat{\beta}_1(.10^3)$	se ₁	$\hat{\beta}_2(.10^{-5})$	se ₂	$\hat{\beta}_3(.10^{-5})$	se ₃	R^2	Obs.
<i>Panel A: S&P 500</i>								
Mar. 16	0.71	(1.60)	1.24	(1.55)	4.78***	(0.83)	0.689	82
Jun. 16	-3.10***	(0.83)	-2.98**	(0.97)	0.67	(0.94)	0.843	82
Sep. 16	-4.89***	(0.63)	-2.14*	(0.77)	3.37***	(0.72)	0.733	82
Dec. 16	-4.38***	(1.35)	2.17*	(0.81)	0.56**	(0.27)	0.545	82
Mar. 17	-8.97***	(0.59)	-0.28	(0.47)	-0.43	(0.76)	0.898	82
Jun. 17	-4.89***	(0.71)	0.18	(0.43)	1.25*	(0.65)	0.706	82
Sep. 17	-2.51***	(0.90)	-1.11*	(0.66)	1.04	(0.80)	0.529	82
Dec. 17	-10.84***	(1.00)	-1.57**	(0.48)	0.59	(1.07)	0.879	82
<i>Panel B: Euro Stoxx 50</i>								
Mar. 16	1.10*	(0.56)	10.24***	(3.43)	6.29***	(1.96)	0.739	82
Jun. 16	-2.37**	(0.94)	12.08	(9.49)	-1.53	(7.31)	0.161	82
Sep. 16	-2.96***	(0.92)	0.07	(1.03)	8.00***	(1.45)	0.502	82
Dec. 16	-3.44***	(1.00)	2.31*	(1.36)	-0.09	(0.77)	0.453	82
Mar. 17	-3.92**	(1.72)	-5.48***	(1.87)	-2.36	(1.83)	0.834	82
Jun. 17	-4.42***	(0.47)	-6.20***	(1.41)	3.22*	(1.70)	0.868	82
Sep. 17	2.05**	(0.82)	2.04***	(0.76)	0.67	(1.63)	0.542	82
Dec. 17	-3.70***	(1.03)	-5.90***	(0.81)	2.92***	(0.71)	0.724	82

Because of Equation (28), the market-implied contango factor is already an implicit function of the spot-futures basis. Meanwhile, the futures risk premium is obviously related the spot-futures basis in a structural way, while being a function of $\tilde{\Phi}$ in Equation (29). As a result, one might rightly suspect that the previous findings are plagued by the endogenous feature of $\tilde{\Phi}(t, T)$ which appears on both sides of our simultaneous equation framework. In Section 5.5, we show how to circumvent the endogeneity of the contango factor by instrumenting $\tilde{\Phi}$ with exogenous variables

driven by the the monetary policy and reflecting the market’s interest rate anticipations.

5.4 The transmission of monetary policy via the contango factor channel

We are now interested in the economic linkage between the contango factor and the interest rate term structure. For this purpose, we have pieced together S&P 500 futures series to reconstitute the S&P 500 “continuous” futures series. More precisely, for each trading day and each futures expiry, the contango factor has first been implied from the futures and underlying cash index closing prices. Second, market-implied contango factors $\tilde{\Phi}(t, T)$ have been weighted by traded volume across futures expiries to get a continuous contango factor $\bar{\Phi}(t)$. As a volume-weighted combination of the various delivery months of the S&P 500 futures, the continuous futures series closely tracks every leading front-month futures series while avoiding sharp discontinuities around rollover periods.

Figure 7a displays $\bar{\Phi}(t)$ as a scatter plot over an extended 2-year period of time (December 2015 to September 2017). Although $\bar{\Phi}(t)$ seems more volatile just after rollover dates, it nevertheless has a low standard deviation of 0.012 around a steady and rising trend-line materialized by its 10-day moving average, which confirms the model’s robustness.

Figure 7. Continuous contango factor implied from S&P 500 futures

These figures plot as a function of time the daily continuous contango factor $\bar{\Phi}(t)$ implied from E-Mini S&P 500 futures against: (a) the LIBOR spot rate, (b) the 30-day Federal Funds futures (FFF) rates. For each S&P 500 futures expiry, daily contango factors $\bar{\Phi}(t, T)$ have been calibrated against the futures and cash index closing prices, and then have been weighted by traded volume across delivery months to construct the daily continuous contango factor $\bar{\Phi}(t)$. S&P 500 futures' costs of carry have been estimated from the cumulated index of annual realized dividend points. Short-term tenors of LIBOR interest rates have provided proxies for the USD money market. Constant volatility parameters have been used for the underlying cash index ($\sigma = 14\%$) and the interest rate term structure ($\nu = 0.1\%$, $a = 0.1$). Data source: Thomson Reuters.

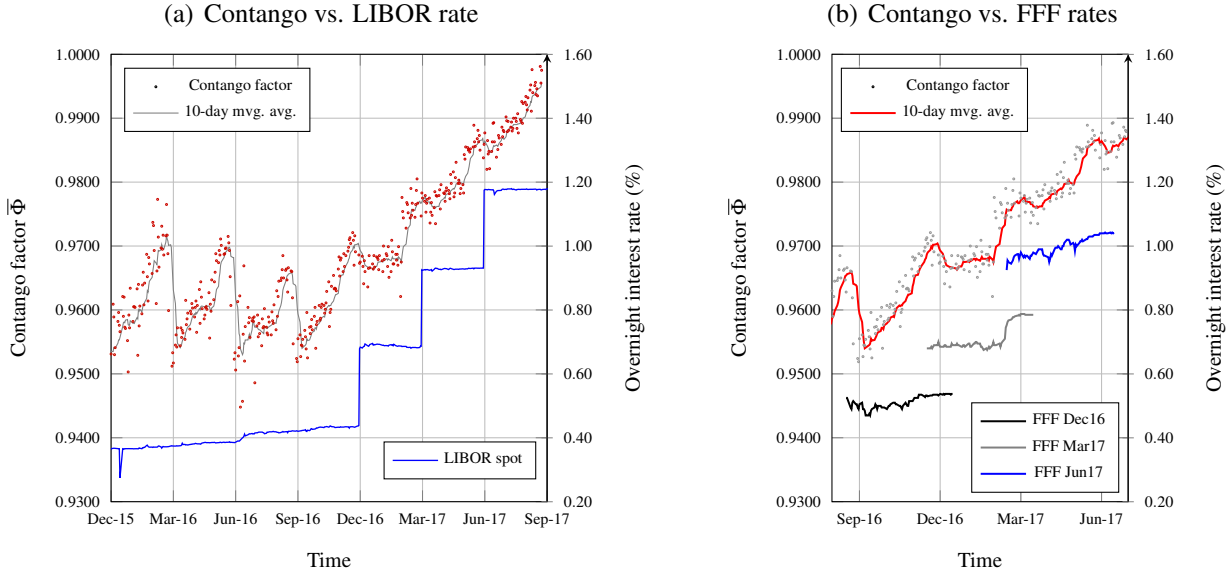


Figure 7a also plots the LIBOR overnight interest rate, which serves as a proxy for the short-term spot interest rate used in our calibrations. Since the US Federal Reserve raised short-term interest rates three times in a row within that time frame,⁸ the plot allows us to measure how the macroeconomic context of rising money market interest rates translates into a higher cost of money on the margin accounts of futures holders. All else being equal, this increase in the cost of money induces an increase in the futures price. Consequently, it has to come with an increase of $\bar{\Phi}(t)$ (meaning a weaker impact of the interest rate growth on the asset value) in order to ensure an equality between the futures price and the expected future price of the cash index. The reason for such a change in the index-rate correlation has to be sought in the macroeconomic context. The

⁸The Federal Open Market Committee (FOMC) raised the federal funds target rate on December 14, 2016 (0.5% to 0.75%), March 15, 2017 (0.75% to 1%) and June 14, 2017 (1% to 1.25%). Incidentally, FOMC meetings were scheduled a few trading days just before oncoming S&P 500 futures expiration dates.

continuing growth of stock markets, coupled with the tightening of the monetary policy, have for consequence that the negative correlation between stock indexes and interest rates tends to become less pronounced and that $\bar{\Phi}(t)$ gets closer to one. We notice that the three rises by the US Federal Reserve were anticipated by S&P 500 futures holders. After each interest rate hike, we observe a reversal in the market-implied contango trend, indicating a temporary widening of the negative index-rate correlation, which may be explained by an overshooting of investors' anticipations.

Figure 7b also plots the closing prices of the three Federal Funds futures (FFF) contracts expiring on the same delivery month as the front-month S&P 500 series. Because the underlying rate of each FFF contract is the 30-day average of the daily *effective* federal funds rates (as calculated and reported by the Federal Reserve Bank of New York over the course of the delivery month), the specificity of FFF contracts is to convey a measure of the market's expectations for potential changes to the federal funds *target* rate over the course of delivery month (e.g., Kuttner, 2001). In line with previous empirical results (De Roon, Nijman, and Veld, 2000), we observe a correlation between the hedging pressure observed on the interest rate futures (FFF rates) and the behavior of the financial futures risk premium (S&P 500 contango factor).

As expirations for FFF contracts follow a monthly cycle,⁹ we choose to overidentify the market-implied contango factor $\tilde{\Phi}(t, T)$ with the three FFF contracts expiring during the front-month period of significant open interest observed on the futures contract. We thus estimate the following multivariate linear regression model:

$$\tilde{\Phi}(t, T) = \gamma_0 + \gamma_1 \cdot \text{FFF}_1(t, T - 2/12) + \gamma_2 \cdot \text{FFF}_2(t, T - 1/12) + \gamma_3 \cdot \text{FFF}_3(t, T) + \varepsilon_t, \quad (30)$$

where $\{\text{FFF}\}_{1 \leq i \leq 3}$ are the three consecutive FFF contracts expiring just before or simultaneously with the futures itself. The null hypothesis is $H_0 : \hat{\gamma}_i = 0$ ($1 \leq i \leq 3$), that is, FFF prices contain no information about the market-implied contango factor, meaning that FFF contracts are weak instruments. As FFF prices evolve as the opposite of their underlying interest rate, H_0 is tested

⁹Expirations for Federal Funds futures contracts occur on the last business day of their delivery months which follow a monthly expiration cycle.

against the alternative that the $\hat{\gamma}_i$'s are significantly negative, meaning that FFF contracts are relevant instruments.

Table 6 reports OLS results of regression (30). With F -statistics far above 10, it appears that not only do FFF prices correlate with the market-implied contango factor, but they can explain a significant portion of its variation. To better assess the explanatory power of FFF instruments, we focus our analysis on the four S&P 500 futures contracts having experienced a sudden rise in the LIBOR overnight interest rate that was triggered by an FOMC policy decision during the 2016-2017 period. For instance, the December 2016 FOMC decision to raise interest rates having been primarily anticipated by the market, the FFF contract maturing in December retains [less](#) explanatory power ($\hat{\gamma}_3 = -0.21$) on the daily market-implied contango factor than the preceding November contract ($\hat{\gamma}_2 = 0.27$). In the same way, the much expected, but still uncertain, June 2017 interest rate hike produced a lot of instability in the interest rate market throughout the spring of 2017. This uncertainty is reflected through the absolute lack of new information conveyed by the May FFF contract ($\hat{\gamma}_2 = 0.01$), and a low explanatory power of the June FFF contract ($\hat{\gamma}_3 = -0.14$). In contrast, the second interest rate hike, decided at the March 15, 2017, FOMC meeting, appears to have remained largely unanticipated by the market until late in the cycle. Because a lot of new information was suddenly conveyed to the market by January and February FFF contracts, a sharp reaction in the market's interest rate expectations can be observed in Figure 7 in late February, a reaction that translates into the contango factor $\Phi^{0,1}$ with the same variability ($\hat{\gamma}_1 = -0.63$ and $\hat{\gamma}_2 = -0.19$). As a result, the March FFF instrument conveys little, if any, new information, and retains much less explanatory power ($\hat{\gamma}_3 = -0.09$).

Table 6. OLS regressions of the S&P 500 contango factor against Federal Funds Futures

This table reports OLS estimates of the slope coefficients in the linear regression:

$$\tilde{\Phi}(t, T) = \gamma_0 + \gamma_1 \cdot \text{FFF}_1(t, T - 2/12) + \gamma_2 \cdot \text{FFF}_2(t, T - 1/12) + \gamma_3 \cdot \text{FFF}_3(t, T) + \varepsilon_t, \quad (29)$$

FOMC signals front-month futures contracts having experienced an effective change in overnight interest rates that was triggered by a FOMC policy decision. t -statistics (not reported) are calculated via robust standard errors (reported in parentheses) corrected for heteroscedasticity and serial correlation. ***, ** and * denote statistical significance at the 1%, 5%, and 10% levels, respectively. Data source: Thomson Reuters.

Contract	FOMC	$\hat{\gamma}_1$	se ₁	$\hat{\gamma}_2$	se ₂	$\hat{\gamma}_3$	se ₃	R^2	F -stat.	Obs.
Mar. 16		0.16	(0.29)	-0.55***	(0.14)	0.28***	(0.07)	0.718	48.4***	61
Jun. 16		-1.23**	(0.51)	0.48**	(0.24)	0.05	(0.11)	0.581	27.3***	63
Sep. 16		-0.30***	(0.06)	0.08	(0.13)	-0.16*	(0.09)	0.509	20.4***	63
Dec. 16	✓	-0.05	(0.05)	0.27***	(0.04)	-0.21***	(0.02)	0.865	128.3***	64
Mar. 17	✓	-0.63***	(0.15)	-0.19*	(0.10)	-0.07***	(0.01)	0.669	38.4***	61
Jun. 17	✓	-0.52*	(0.27)	0.01	(0.15)	-0.14***	(0.03)	0.758	61.7***	63
Sep. 17		0.04	(0.22)	-0.27	(0.17)	0.62***	(0.11)	0.694	44.6***	63
Dec. 17	✓	-0.30	(0.20)	0.10	(0.15)	-0.13***	(0.03)	0.634	34.7***	64

5.5 The impact of the monetary policy on the backwardation/contango regime

If the market's expectations about monetary policy are well known to have a direct effect on the stock market (e.g., Bernanke and Kuttner, 2005), this effect should translate altogether to stock index futures, and thus cancel out when considering the spot-futures basis or the futures risk premium. Put differently, there should be no direct transmission of the exogenous anticipations contained in FFF prices to the S&P 500 futures risk premium, except through the channel of the whole interest rate term structure that is already factored in the day-to-day calibration of the implied contango factor. In consequence, FFF prices lack compelling reasons for being considered exogenous regressors of the futures risk premium. Against this backdrop, it seems justified to use the exogeneity of FFFs instruments as instrumental variables for investigating the link between the market-implied contango factor and the contemporaneous risk premium in the futures market.

To assess the impact of monetary policy on the backwardation versus contango regime of financial futures, we now estimate regression (29) by the two-stage least squares (TSLS) method in which FFFs are used as instrumental variables. In this setting, the TSLS second stage has to be restricted to the 3-month time period over which the S&P 500 futures of maturity T is the front-

month contract (64 daily observations). Meanwhile, the TSLS first stage tests the correlation with the FFF contract maturing in the same delivery month, $FFF(t, T)$, over the same time period.

Table 7 reports estimates for the TSLS second stage. In the case of S&P 500 futures, these results amplify the preliminary findings of Table 5. Slope coefficients of regression (29) keep their signs and magnitudes via TSLS estimation. More importantly, the statistical significance of the slope coefficient $\hat{\beta}_1$ uniformly increases via TSLS estimation compared to OLS estimation. The fact that regression (29) performs better by instrumenting the contango factor with Federal Funds futures provides reliable evidence that the contango factor plays some role in crystallizing the market's interest rates expectations into financial futures risk premia. The sensitivity of the futures risk premium to a rise in short-term interest rates can be estimated as:

$$\frac{\Delta F}{\Delta r} = \frac{\Delta F}{\Delta \tilde{\Phi}} \cdot \frac{\Delta \tilde{\Phi}}{\Delta r} \approx (-\hat{\beta}_1) \times (-\hat{\gamma}_3) = \hat{\beta}_1 \hat{\gamma}_3.$$

For a standard interest rate hike of 25 basis points triggered by the US Federal Reserve, we find positive sensitivities (i.e., increases in futures prices leading to deflated futures risk premium) ranging from +2 index points (December 2016, June 2017) to +17 points (March 2017).

Table 7. TSLS regressions of the S&P 500 futures risk premium against the contango factor

This table reports TSLS estimates of the slope coefficients in the linear regressions:

$$F(T, T) - F(t, T) = \beta_0 + \beta_1 \cdot \tilde{\Phi}(t, T) + \beta_2 \cdot \text{OpInt}(t) + \beta_3 \cdot \text{Vol}(t) + \varepsilon_t, \quad (29)$$

with FFFs as instrumental variables. FOMC signals front-month futures contracts having experienced an effective change in overnight interest rates that was triggered by a FOMC policy decision. F is the first-stage F -statistic testing for instrument weakness. HW is the second-stage Hausman-Wu statistic testing for endogeneity of regressors. J is the second-stage over-identification Sargan statistic testing for the instrument set validity. t -statistics (not reported) are calculated via robust standard errors (reported in parentheses) corrected for heteroscedasticity. ***, ** and * denote statistical significance at the 1%, 5%, and 10% levels, respectively.

Contract	FOMC	$\hat{\beta}_1(.10^3)$	se ₁	$\hat{\beta}_2(.10^{-5})$	se ₂	$\hat{\beta}_3(.10^{-5})$	se ₃	F	HW	J	Obs.
Mar. 16		2.17***	(0.61)	5.41***	(1.98)	5.56***	(0.70)	53.6***	29.1***	0.5	61
Jun. 16		-2.93***	(0.45)	-1.33***	(0.32)	2.23***	(0.83)	27.4***	2.0	3.9	63
Sep. 16		-11.68***	(1.80)	-6.01***	(1.24)	0.58	(1.26)	11.5***	122.1***	0.5	63
Dec. 16	✓	-4.17***	(0.49)	5.25***	(0.69)	0.80**	(0.39)	124.2***	5.8**	11.2***	64
Mar. 17	✓	-10.86***	(0.74)	-0.70	(0.68)	1.40	(0.88)	31.7***	7.8*	6.9**	61
Jun. 17	✓	-5.76***	(0.56)	0.96**	(0.43)	1.76***	(0.55)	59.1***	27.3***	8.6**	63
Sep. 17		-3.22***	(0.47)	0.70*	(0.41)	2.55***	(0.46)	50.0***	14.4***	5.5*	63
Dec. 17	✓	-13.67***	(0.95)	-1.56	(0.96)	1.20	(0.10)	41.5***	63.0***	13.8	64

Table 7 also reports TSLS diagnostic tests. First, first-stage F -statistics testing for instrument weakness are well over 10 and consistently significant at the 1% level. These numbers confirm the preliminary results of Table 6 and provide reliable evidence that FFF contracts are relevant for instrumenting $\tilde{\Phi}$. Second, we also check the endogeneity of the market-implied contango factor. In six cases out of eight, the Hausman-Wu statistic rejects the null hypothesis that $\tilde{\Phi}$ is exogenous at the 5% significance level. This high level of rejection confirms that the TSLS first stage essentially succeeds in isolating the variation in $\tilde{\Phi}$ that is correlated with the error term in specification (29), thereby validating the TSLS approach. Finally, for a few S&P 500 futures expiries, we notice that the Sargan test for over-identification rejects the validity of the FFF instrument set. Indeed, Table 6 shows that some FFF contracts may happen to be irrelevant, especially in the beginning of the front-month cycle. For example, if we remove the October 2016 FFF contract from the list of instruments for the December 2016 S&P 500 contract, estimates of Table 7 remain unchanged while the over-identifying test no longer rejects at the 1% level.

6. Conclusions

In this article, we propose a modeling framework applicable to financial futures contracts and their derivatives. The key parameter of the model is the contango factor Φ , which captures the propensity of the underlying asset price to co-evolve with the money market interest rate. The linkage between the money market funding rate and the underlying asset price is stochastic and endogenous, in consistency with investors' arbitrage strategies.

Our model explicitly captures the impact of the marking-to-market feature of futures, that is, the forward-futures price difference predicted by Cox, Ingersoll, and Ross's (1981) theory. As a result, the terminal distributions of the underlying asset price and its futures price are shifted according to the endogenous asset-rate covariance. Crystallized through the daily margin calls of the futures contract, a contango factor higher (lower) than one leads to a more expensive (cheaper) futures price and a contango (normal backwardation) regime. Akin to the implied volatility of option contracts, the market-implied contango factor provides market participants with a universal gauge of the level of contango. This new measure is consistent across futures markets and maturities. Besides the cost of carry and the premium for bearing the price fluctuation risk originally put forward by Keynes (1930), we find that the backwardation/contango regime also depends on the asset-rate covariance captured in our model by the parameter Φ . The introduction of this new factor paves the way for new empirical studies.

Our numerical simulations indicate that the divergence from the traditional cost-of-carry model of financial futures can be significant, with price deviations larger than 1%, even for short-term futures contracts. Our study confirms that the dynamics of interest rates—to which previous studies have only alluded—play a major role in the optimal conversion policy of American-style futures options. Our findings underscore, in particular, the significant lowering impact of the contango factor on their optimal exercise boundary.

Finally, estimating the model on S&P 500 and Euro Stoxx 50 historical data confirms the role of the market's interest rate expectations in the formation of financial futures risk premia at low

frequencies. We find that the recent interest rate rises decided by the US Federal Reserve had deflating impacts of up to 17 index points on the daily S&P 500 futures risk premium. Our empirical tests thus highlight the impact of the monetary policy on the backwardation/contango regime of financial futures, thereby shedding new light on Keynes's (1930) theory of normal backwardation.

Appendix A. Proof of Proposition 1

We proceed to prove the core result (a) of Proposition 1 by the absence of arbitrage principle. We assume that $\Phi_n^{0,1}$ is known. Our objective is to establish a relationship between $\Phi_n^{0,1}$ and $\Phi_n^{0,-1}$. We consider that the initial node is $(n; 0, 0)$, noting that the demonstration would be exactly the same with any other value of the economic factors index i . We assume that at the next time-step $n + 1$, the short rate index attains levels 1, 0 or -1 . A portfolio $P := \{-Q_S S_n, Q_B, 1\}$ with the following features is set up:

1. selling a quantity Q_S of the underlying asset S_n , where S_n is the asset price at date n ;
2. owing a $\$Q_B$ discount bond, maturing at the end of period $n + 2$;
3. owing a $\$1$ discount bond, maturing at the end of period $n + 3$.

The portfolio value at step n is as follows:

$$P(n; 0, 0) = -Q_S S_n + \frac{Q_B}{1 + R_{n,0}(2)} + \frac{1}{1 + R_{n,0}(3)}, \quad (\text{A1})$$

where $R_{n,0}(j)$ is at node $(n;0,0)$ the yield of a discount bond maturing in j periods¹⁰. At step $n+1$, the portfolio may have ten different values which can easily be formulated, of which three are of particular interest:

$$\begin{cases} P(n+1;0,1) = -Q_s S_n \Phi_n^{0,1} \frac{1+r_0}{1+r_1} + \frac{Q_B}{1+r_1} + \frac{1}{1+R_{n+1,1}(2)}, \\ P(n+1;0,0) = -Q_s S_n \Phi_n^{0,0} + \frac{Q_B}{1+r_0} + \frac{1}{1+R_{n+1,0}(2)}, \\ P(n+1;0,-1) = -Q_s S_n \Phi_n^{0,-1} \frac{1+r_0}{1+r_{-1}} + \frac{Q_B}{1+r_{-1}} + \frac{1}{1+R_{n+1,-1}(2)}. \end{cases} \quad (\text{A2})$$

To immunize the portfolio P against interest rate movements, we first solve for the asset quantity Q_s^* and for the discount bond quantity Q_B^* which equalize portfolio values $P(n+1;0,1)$, $P(n+1;0,0)$ and $P(n+1;0,-1)$. The solution of the corresponding linear system is:

$$Q_s^* = \frac{-1}{S_n} \cdot \frac{\frac{1+r_1}{1+r_0} \Delta_R^{0,1} + \frac{1+r_{-1}}{1+r_0} \Delta_R^{0,-1}}{\Phi_n^{0,1} - \Phi_n^{0,-1} - \frac{2\Delta r}{1+r_0} \Phi_n^{0,0}}, \quad (\text{A3})$$

and:

$$Q_B^* = \frac{\Phi_n^{0,1} - \frac{1+r_1}{1+r_0} \Phi_n^{0,0}}{\Phi_n^{0,1} - \Phi_n^{0,-1} - \frac{2\Delta r}{1+r_0} \Phi_n^{0,0}} \cdot \frac{\frac{1+r_1}{1+r_0} \Delta_R^{0,1} + \frac{1+r_{-1}}{1+r_0} \Delta_R^{0,-1}}{\Delta_r^{1,0}} + \frac{\Delta_R^{0,1}}{\Delta_r^{1,0}}, \quad (\text{A4})$$

where Δr is the interest rate tick size, and where:

$$\Delta_r^{1,0} := \frac{1}{1+r_1} - \frac{1}{1+r_0}, \quad (\text{A5})$$

¹⁰ $R_{n,0}(j)$ is not expressed in annual rates but over j periods. In particular, $1/(1+R_{n,k}(2))$ is the value as seen at node $(n;k,i)$ of a 2-period discount bond maturing at time $(n+2)\Delta t$, and is known analytically in the case of the Hull and White (1993) model:

$$\frac{1}{1+R_{n,k}(2)} = \frac{e^{(-h_n + ar_k + v^2 \Delta t / 2) \Delta t^2}}{(1+r_k)^2}.$$

In the same way, $1/(1+R_{n,k}(3))$ is the value as seen at node $(n;k,i)$ of a 3-period discount bond maturing at time $(n+3)\Delta t$, and may be obtained as:

$$\frac{1}{1+R_{n,k}(3)} = \frac{1}{1+r_k} \left[\frac{p_n^{k,k+1}}{1+R_{n+1,k+1}(2)} + \frac{p_n^{k,k}}{1+R_{n+1,k}(2)} + \frac{p_n^{k,k-1}}{1+R_{n+1,k-1}(2)} \right].$$

and:

$$\Delta_R^{0,\pm 1} := \frac{1}{1 + R_{n+1,0}(2)} - \frac{1}{1 + R_{n+1,\pm 1}(2)}. \quad (\text{A6})$$

Thus, the portfolio $P^* := \{-Q_s^* S_n, Q_B^*, 1\}$ is immunized at step $n + 1$ against any movement in the interest rate in case the parameters of the asset are stable. As a result we have:

$$P^*(n + 1; 0, 0) = P^*(n + 1; 0, 1) = P^*(n + 1; 0, -1). \quad (\text{A7})$$

Note that P^* depends explicitly on $\Phi_n^{0,1}$ and $\Phi_n^{0,-1}$ through the quantities Q_s^* and Q_B^* .

We now consider two other portfolios:

$$P_d^* := \left\{ -\frac{Q_s^* S_n}{d}, Q_B^*, 1 \right\} \quad \text{and} \quad P_u^* := \left\{ -\frac{Q_s^* S_n}{u}, Q_B^*, 1 \right\}, \quad (\text{A8})$$

again depending explicitly on $\Phi_n^{0,1}$ and $\Phi_n^{0,-1}$. Because of the symmetry in the composition of P_d^* and P_u^* , the portfolio P_d^* (resp. P_u^*) is immunized at step $n + 1$ against any movement in the interest rate in case the parameters of the asset are bearish (resp. bullish). As a result we have:

$$P_d^*(n + 1; -1, 0) = P_d^*(n + 1; -1, 1) = P_d^*(n + 1; -1, -1), \quad (\text{A9})$$

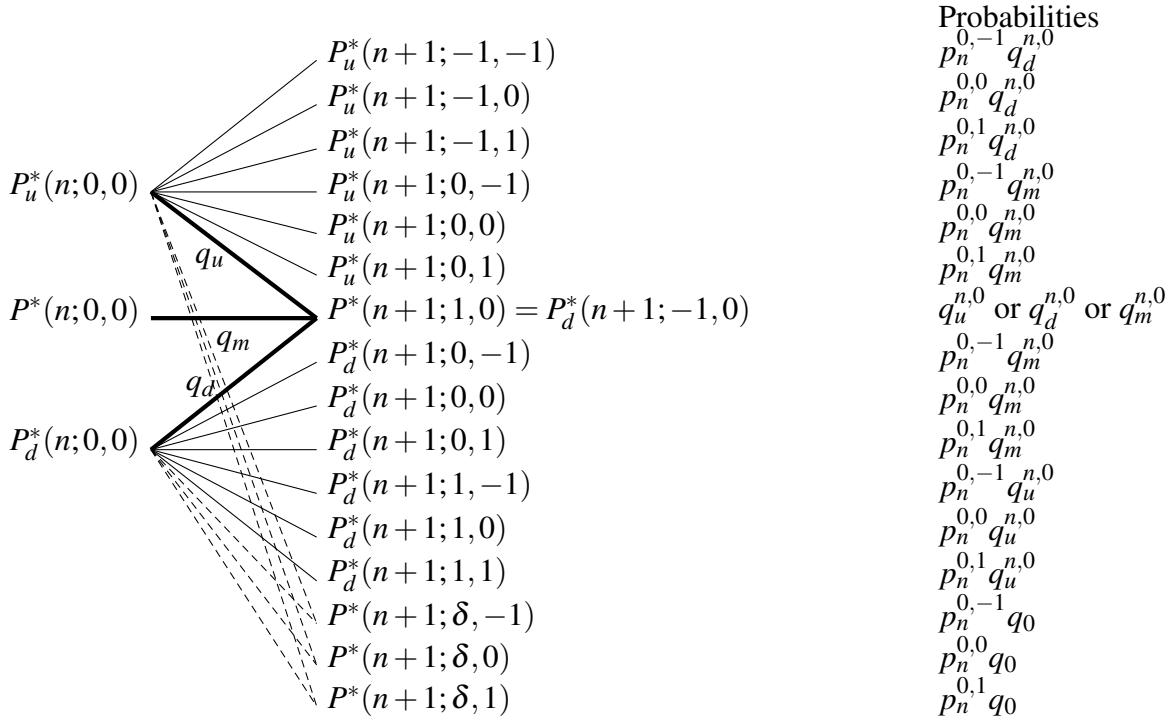
and:

$$P_u^*(n + 1; 1, 0) = P_u^*(n + 1; 1, 1) = P_u^*(n + 1; 1, -1). \quad (\text{A10})$$

By construction of portfolios P^* and P_d^* (resp. P_u^*), it may also be noted that:

$$P_d^*(n + 1; -1, 0) = P^*(n + 1; 0, 0) \quad (\text{resp.} \quad P_u^*(n + 1; 1, 0) = P^*(n + 1; 0, 0)). \quad (\text{A11})$$

Therefore, we obtain the following graph:



where $q_u^{n,0}$ (resp. $q_m^{n,0}$, $q_d^{n,0}$) is the probability of a bullish (resp. stable, bearish) evolution of the asset's economic factors, q_0 is the probability of a jump of the asset in the default state δ , $p_n^{0,1}$ (resp. $p_n^{0,0}$, $p_n^{0,-1}$) is the probability of an upward (resp. stable, downward) evolution of the interest rate.

When moving backwards, this tree enables the value of the various portfolios at the previous step to be set. In particular, at step n , the anticipated value of portfolio $P^*(n+1;0,0)$ may be established. Under the assumption of fully absorbing default ($\delta \equiv 0$), only portfolios $P^*(n;0,0)$, $P_u^*(n;0,0)$ and $P_d^*(n;0,0)$ make access to portfolio $P^*(n+1;0,0)$ possible at step $n+1$. The investor's anticipated value at step n of this portfolio is therefore equal to:

- $P_u^*(n;0,0)$ in case the asset price between n and $n+1$ is bullish and whatever evolution the interest rate takes (which occurs with probability $q_u^{n,0}$);
- $P_d^*(n;0,0)$ in case the asset price between n and $n+1$ is bearish and whatever evolution the interest rate takes (which occurs with probability $q_d^{n,0}$);
- $P^*(n;0,0)$ in case the asset price between n and $n+1$ is stable and whatever evolution the interest rate takes (which occurs with probability $q_m^{n,0}$).

Thus, the investor's anticipated value of the portfolio $P^*(n+1;1,0)$ is as follows:

$$q_u^{n,0} \cdot P_u^*(n;0,0) + q_d^{n,0} \cdot P_d^*(n;0,0) + q_m^{n,0} \cdot P^*(n;0,0). \quad (\text{A12})$$

We now consider the investment strategy which consists of purchasing at step n a discount bond, maturing at $n+1$, for an amount of $P^*(n+1;0,0)/(1+r_0)$, r_0 being the interest rate which is applied between steps n and $n+1$. Such an investment, carried out over the period n , leads to the value $P^*(n+1;0,0)$ at step $n+1$. Without any arbitrage opportunity, this investment strategy must be equivalent to the strategy which consists of purchasing the portfolio defined by Equation (A12). The following equation must then be satisfied:

$$q_u^{n,0} \cdot P_u^*(n;0,0) + q_d^{n,0} \cdot P_d^*(n;0,0) + q_m^{n,0} \cdot P^*(n;0,0) = \frac{P^*(n+1;0,0)}{1+r_0}. \quad (\text{A13})$$

As shown earlier, the quantities P^* , P_u^* and P_d^* in Equation (A13) depend explicitly on $\Phi_n^{0,1}$ and $\Phi_n^{0,-1}$. By substituting Equations (A3) and (A4) into Equation (A13), the value that is being sought for here, that is, $\Phi_n^{0,-1}$, can be calculated explicitly from the pure diffusion parameters, the interest rate term structure $R_{n,0}(j)$ and the pre-determined parameter $\Phi_n^{0,1}$.

To get the explicit expression of $\Phi_n^{0,-1}$ as a function of $\Phi_n^{0,1}$ and $\Phi_n^{0,0}$, we first substitute Equation (A1) into (A13):

$$\begin{aligned} & q_u^{n,0} \left(-\frac{Q_s^* S_n}{u} + \frac{Q_B^*}{1+R_{n,0}(2)} + \frac{1}{1+R_{n,0}(3)} \right) + q_d^{n,0} \left(-\frac{Q_s^* S_n}{d} + \frac{Q_B^*}{1+R_{n,0}(2)} + \frac{1}{1+R_{n,0}(3)} \right) \\ & + q_m^{n,0} \left(-Q_s^* S_n + \frac{Q_B^*}{1+R_{n,0}(2)} + \frac{1}{1+R_{n,0}(3)} \right) = \frac{-Q_s^* S_n + \frac{Q_B^*}{1+r_0} + \frac{1}{1+R_{n+1,0}(2)}}{1+r_0}, \end{aligned} \quad (\text{A14})$$

which can be reorganized as:

$$-\left(\frac{q_u^{n,0}}{u} + \frac{q_d^{n,0}}{d} + q_m^{n,0} - \frac{1}{1+r_0} \right) Q_s^* S_n + \left(q_u^{n,0} + q_d^{n,0} + q_m^{n,0} - \frac{1+R_{n,0}(2)}{(1+r_0)^2} \right) \frac{Q_B^*}{1+R_{n,0}(2)}$$

$$+ \frac{q_u^{n,0} + q_d^{n,0} + q_m^{n,0}}{1 + R_{n,0}(3)} - \frac{1}{(1+r_0)(1+R_{n+1,0}(2))} = 0. \quad (\text{A15})$$

Second, we can simplify $q_u^{n,0} + q_d^{n,0} + q_m^{n,0}$ by 1 and substitute Equations (A3) and (A4) for Q_s^* and Q_B^* . Multiplying by the denominator of Q_s^* leads to:

$$\begin{aligned} & \left(\frac{q_u^{n,0}}{u} + \frac{q_d^{n,0}}{d} + q_m^{n,0} - \frac{1}{1+r_0} \right) \left(\frac{1+r_1}{1+r_0} \Delta_R^{0,1} + \frac{1+r_{-1}}{1+r_0} \Delta_R^{0,-1} \right) \\ & + \left(\Phi_n^{0,1} - \frac{1+r_1}{1+r_0} \Phi_n^{0,0} \right) \left(\frac{1}{1+R_{n,0}(2)} - \frac{1}{(1+r_0)^2} \right) \frac{\frac{1+r_1}{1+r_0} \Delta_R^{0,1} + \frac{1+r_{-1}}{1+r_0} \Delta_R^{0,-1}}{\Delta_r^{1,0}} \\ & + \left(\Phi_n^{0,1} - \Phi_n^{0,-1} - \frac{2\Delta r}{1+r_0} \Phi_n^{0,0} \right) \left(\frac{1}{1+R_{n,0}(2)} - \frac{1}{(1+r_0)^2} \right) \frac{\Delta_R^{0,1}}{\Delta_r^{1,0}} \\ & + \left(\Phi_n^{0,1} - \Phi_n^{0,-1} - \frac{2\Delta r}{1+r_0} \Phi_n^{0,0} \right) \left(\frac{1}{1+R_{n,0}(3)} - \frac{1}{(1+r_0)(1+R_{n+1,0}(2))} \right) = 0. \quad (\text{A16}) \end{aligned}$$

It is now clear on Equation (A16) that $\Phi_n^{0,-1}$ may be expressed as a linear recursion of $\Phi_n^{0,1}$ and $\Phi_n^{0,0}$.

To prove item (b) of Proposition 1, we note that the same line of reasoning applies more generally at any level k of the interest rate. For negative indexes of the interest rate level ($k \leq 0$), the linear recursive relationship to be obtained is as follows:

$$\Phi_n^{k,k-1} = \frac{A_{n,k}}{C_{n,k} + D_{n,k}} + \frac{B_{n,k} + C_{n,k} + D_{n,k}}{C_{n,k} + D_{n,k}} \cdot \Phi_n^{k,k+1} - \frac{\frac{1+r_{k+1}}{1+r_k} B_{n,k} + \frac{2\Delta r}{1+r_k} (C_{n,k} + D_{n,k})}{C_{n,k} + D_{n,k}} \cdot \Phi_n^{k,k}, \quad (\text{A17})$$

where:

$$A_{n,k} := \left(\frac{q_u^{n,k}}{u} + \frac{q_d^{n,k}}{d} + q_m^{n,k} - \frac{1}{1+r_k} \right) \left(\frac{1+r_{k+1}}{1+r_k} \Delta_R^{k,k+1} + \frac{1+r_{k-1}}{1+r_k} \Delta_R^{k,k-1} \right), \quad (\text{A18})$$

$$B_{n,k} := \left(\frac{1}{1+R_{n,k}(2)} - \frac{1}{(1+r_k)^2} \right) \frac{\frac{1+r_{k+1}}{1+r_k} \Delta_R^{k,k+1} + \frac{1+r_{k-1}}{1+r_k} \Delta_R^{k,k-1}}{\Delta_r^{k+1,k}}, \quad (\text{A19})$$

$$C_{n,k} := \left(\frac{1}{1+R_{n,k}(2)} - \frac{1}{(1+r_k)^2} \right) \frac{\Delta_R^{k,k+1}}{\Delta_r^{k+1,k}}, \quad (\text{A20})$$

$$D_{n,k} := \frac{1}{1 + R_{n,k}(3)} - \frac{1}{(1 + r_k)(1 + R_{n+1,k}(2))}, \quad (\text{A21})$$

$$\Delta_r^{k+1,k} := \frac{1}{1 + r_{k+1}} - \frac{1}{1 + r_k}, \quad (\text{A22})$$

$$\Delta_R^{k,k\pm 1} := \frac{1}{1 + R_{n+1,k}(2)} - \frac{1}{1 + R_{n+1,k\pm 1}(2)}. \quad (\text{A23})$$

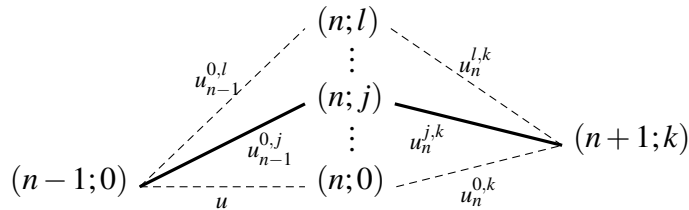
For positive indexes of the interest rate level ($k \geq 0$), the linear recursive relationship to be obtained is given by:

$$\Phi_n^{k,k+1} = \frac{-A_{n,k}}{B_{n,k} + C_{n,k} + D_{n,k}} + \frac{C_{n,k} + D_{n,k}}{B_{n,k} + C_{n,k} + D_{n,k}} \cdot \Phi_n^{k,k-1} + \frac{\frac{1+r_{k+1}}{1+r_k} B_{n,k} + \frac{2\Delta_r}{1+r_k} (C_{n,k} + D_{n,k})}{B_{n,k} + C_{n,k} + D_{n,k}} \cdot \Phi_n^{k,k}. \quad (\text{A24})$$

To prove item (c) of Proposition 1, we note that once $\Phi_n^{k,k} = 1$ and $\Phi_n^{0,1}$ are known, it is possible to determine all the values of $\Phi_n^{0,k}$ for positive indexes ($k > 0$) of the short rate in a recursive way, thanks to Equations (A24) and (12). In the same way, it is possible to determine all the values of $\Phi_n^{0,k}$ for negative indexes ($k < 0$) of the short rate in a recursive way, thanks to Equations (A17) and (13).

Appendix B. Proof of Proposition 2

- Owing to the symmetry of the model, here we examine only the case of a bullish evolution in the fundamental economic factors of the underlying asset. Considering the following two-node path appearing in bold:



the recombining feature of the lattice (Assumption 5) is equivalent to:

$$\forall i, j, k, l, \quad u_{n-1}^{i,j} \cdot u_n^{j,k} = u_{n-1}^{i,l} \cdot u_n^{l,k}. \quad (\text{B1})$$

This condition must be checked for $i = 0$ and $l = 0$ (the null index indicates the interest rate at the origin of the lattice). Consequently, a necessary condition for meeting recombination is as follows:

$$\forall j, k, \quad u_n^{j,k} = u \cdot u_n^{0,k} / u_{n-1}^{0,j}, \quad (\text{B2})$$

which is the first part of Proposition 2.

- Reciprocally, to show that condition (B2) is sufficient for meeting recombination, we use (B2) to write the product $u \cdot u_n^{0,k}$ in two different ways, first as $u_{n-1}^{0,j} \cdot u_n^{j,k}$, and second as $u_{n-1}^{0,l} \cdot u_n^{l,k}$. Multiplying by u , we get:

$$u \cdot u_{n-1}^{0,j} \cdot u_n^{j,k} = u \cdot u_{n-1}^{0,l} \cdot u_n^{l,k}, \quad (\text{B3})$$

which enables to apply (B2) once again on the first two terms:

$$u_{n-2}^{0,i} \cdot u_{n-1}^{i,j} \cdot u_n^{j,k} = u_{n-2}^{0,i} \cdot u_{n-1}^{i,l} \cdot u_n^{l,k}, \quad (\text{B4})$$

which leads to condition (B1) after simplifying by $u_{n-2}^{0,i}$.

- For a bearish evolution of the fundamental economic factors, we would obtain a symmetrical relation:

$$\forall j, k, \quad d_n^{j,k} = d \cdot d_n^{0,k} / d_{n-1}^{0,j}. \quad (\text{B5})$$

which is the last equation in Proposition 2.

- In case of a stable evolution of the fundamental economic factors, we notice that Equation (B2) implies the following relationship:

$$m_n^{j,k} = m \cdot \frac{m_n^{0,k}}{m_{n-1}^{0,j}} = \frac{\Phi_n^{0,k}}{\Phi_{n-1}^{0,j}}, \quad (\text{B6})$$

which ensures a convergence of paths within the lattice.

Appendix C. Proof of Proposition 3

The result comes directly from the observation that at node $(n; i, j)$, in the case where the interest rate stays at the level j between time-steps n and $n + 1$, the marginal probabilities should match the first two moments of the underlying asset price diffusion. First, the probabilities sum to one:

$$q_u^{n,j} + q_m^{n,j} + q_d^{n,j} + q_0 = 1. \quad (\text{C1})$$

Second, the mean of the discrete distribution is equal to the mean of the continuous lognormal distribution:

$$q_u^{n,j} u_n^{j,j} S_n^{i,j} + q_d^{n,j} d_n^{j,j} S_n^{i,j} + q_m^{n,j} m_n^{j,j} S_n^{i,j} = M_j S_n^{i,j}, \quad (\text{C2})$$

where the evolution coefficients within the lattice at level j are given by Proposition 2:

$$\begin{cases} u_n^{j,j} = u \cdot m_n^{j,j}, \\ m_n^{j,j} = \Phi_n^{0,j} / \Phi_{n-1}^{0,j}, \\ d_n^{j,j} = d \cdot m_n^{j,j}, \end{cases} \quad (\text{C3})$$

and where the conditional mean of the asset price process is driven by the growth rate in the money market account:

$$M_j := \frac{\mathbb{E}[S_{n+1}^{i,j} | S_n^{i,j}]}{S_n^{i,j}} = (1 + r_j - q)\Delta t. \quad (\text{C4})$$

Third, the variance of the discrete distribution is equal to the variance of the continuous distribution:

$$q_u^{n,j} (S_n^{i,j})^2 [(u_n^{j,j})^2 - M_j^2] + q_d^{n,j} (S_n^{i,j})^2 [(d_n^{j,j})^2 - M_j^2] + q_m^{n,j} (S_n^{i,j})^2 [(m_n^{j,j})^2 - M_j^2] = V_j (S_n^{i,j})^2, \quad (\text{C5})$$

where the conditional variance is given by:

$$V_j := \frac{\text{Var} \left[S_{n+1}^{i,j} | S_n^{i,j} \right]}{\left(S_n^{i,j} \right)^2} = \sigma^2 \Delta t. \quad (\text{C6})$$

Dividing the second equation (C2) by $S_n^{i,j}$ and the third equation (C5) by $\left(S_n^{i,j} \right)^2$, and reorganizing (C5), we obtain:

$$\begin{cases} q_u^{n,j} + q_m^{n,j} + q_d^{n,j} = 1 - q_0, \\ q_u^{n,j} m_j u + q_d^{n,j} m_j d + q_m^{n,j} m_j = M_j, \\ q_u^{n,j} m_j^2 u^2 + q_d^{n,j} m_j^2 d^2 + q_m^{n,j} m_j^2 = V_j + M_j^2 (1 - q_0), \end{cases} \quad (\text{C7})$$

where $m_j := m_n^{j,j}$. The first equation can be used to remove $q_m^{n,j} m_j$ from the last two equations, and we are left with solving the following linear system:

$$\begin{cases} q_u^{n,j} m_j (u - 1) + q_d^{n,j} m_j (d - 1) = M_j - m_j (1 - q_0), \\ q_u^{n,j} m_j^2 (u^2 - 1) + q_d^{n,j} m_j^2 (d^2 - 1) = V_j + M_j^2 (1 - q_0) - m_j^2 (1 - q_0), \end{cases} \quad (\text{C8})$$

the solution of which is given by:

$$\begin{cases} q_u^{n,j} = \frac{u(V_j + M_j^2(1 - q_0) - m_j M_j) - m_j(M_j - m_j(1 - q_0))}{m_j^2(u - 1)(u^2 - 1)}, \\ q_d^{n,j} = \frac{u^2(V_j + M_j^2(1 - q_0) - m_j M_j) - u^3 m_j(M_j - m_j(1 - q_0))}{m_j^2(u - 1)(u^2 - 1)}. \end{cases} \quad (\text{C9})$$

Note that in the non-hybrid, non-defaultable case (i.e., $j = 0$, $m_j = 1$ and $q_0 = 0$) we recover the standard trinomial transition probabilities from Boyle (1988):

$$\begin{cases} q_u = \frac{(V + M^2 - M)u - (M - 1)}{(u - 1)(u^2 - 1)}, \\ q_d = \frac{u^2(V + M^2 - M) - u^3(M - 1)}{(u - 1)(u^2 - 1)}. \end{cases} \quad (\text{C10})$$

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